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Vertex-Distinguishing Edge Coloring of Fibonacci Cubes

Tigran Petrosyan
Russian-Armenian Slavonic University
Yerevan, Armenia
email: tigran.petrosyan@student.rau.am

Petros Petrosyan
Yerevan State University
Yerevan, Armenia
email: petros_petrosyan@ysu.am

***Abstract*—A proper edge coloring of a graph G is a mapping $\phi : E(G) \longrightarrow \mathbb{Z}_{\geq 0}$ such that $\phi(e) \neq \phi(e')$ for every pair of adjacent edges e and e' in G. A proper edge coloring ϕ of a graph G is vertex-distinguishing if for any different vertices $u, v \in V(G)$, $S(u,\phi) \neq S(v,\phi)$, where $S(v,\phi) = \{\phi(e) \mid e = uv \in E(G)\}$. The minimum number of colors required for a vertex-distinguishing proper edge coloring of a simple graph G is denoted by $\chi'_{vd}(G)$. Fibonacci cubes F_n can be obtained from hypercubes Q_n by removing vertices x with consecutive 1s. In this paper we show that $n \leq \chi'_{vd}(F_n) \leq n+1$.**

***Keywords*— Edge coloring, vertex-distinguishing coloring, chromatic index, Fibonacci cube**

I. Introduction

All graphs considered in this paper are finite and simple. We denote by $V(G)$ and $E(G)$ the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d(v)$ and the maximum degree of vertices in G by $\Delta(G)$. An induced matching in a graph G is an induced subgraph of G that forms a matching.

An *edge coloring* of a graph G is a mapping $\phi : E(G) \to \mathbb{N}$. ϕ is called *strong* if each color class is an induced matching. The minimum number of colors required for a strong edge coloring of a graph G is called *strong chromatic index* of graph G and denoted by $\chi'_s(G)$.

The $VDP-coloring$ has been considered in many papers. It was introduced and studied by Burris and Schelp in [1,2] and, independently, as observability of a graph, by Cerný et al., Hornák and Soták [3]. In [4,5], the $VDP-coloring$ is also computed for some families of graphs, such as complete graphs K_n, bipartite complete graphs $K_{m,n}$, split graphs, paths P_n, and cycles C_n. The following results have been proved by Burris and Schelp [2].

Theorem 1: Let n be any natural number. Then

$$\chi'_{vd}(K_n) = \begin{cases} n & \text{if } \ n \ is \ odd; \\ n+1 & \text{if } \ n \ is \ even. \end{cases}.$$

Theorem 2: Let m and n be any natural numbers. Then

$$\chi'_{vd}(K_{m,n}) = \begin{cases} n+1 & \text{if} \quad n > m \geq 2; \\ n+2 & \text{if} \quad n = m \geq 2. \end{cases}$$

The original motivation of study is generalizing results for $VDP-coloring$.

II. Main result

Theorem 3: Let n be any natural number greater than 3. Then

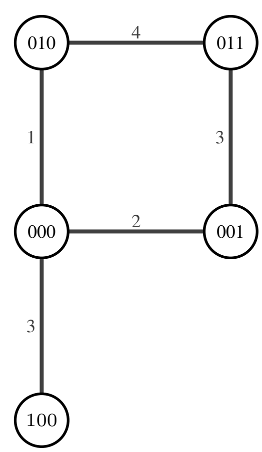
$$n \leq \chi'_{vd}(F_n) \leq n+1.$$

Proof:

In order to prove the lower bound let us consider all the edges that are adjacent to vertex $x_0 = 00...0$. There are n edges adjacent to vertex x_0. They should all be colored with different colors. We get $\chi'_s(F_n) \geq n$. Let us construct a vertex-distinguishing edge coloring with $n+1$ colors ϕ of F_n by induction on n.

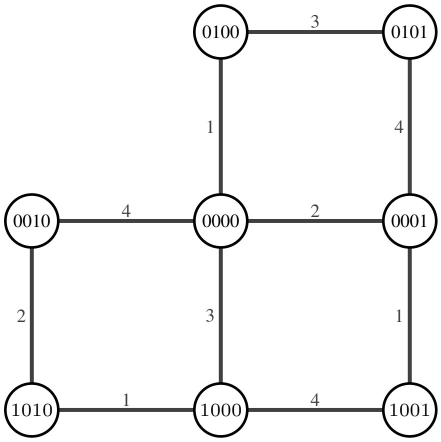
For base cases $n = 3$:

$$\phi(e) = \begin{cases} 1 & e \in \{(000, 010)\} \\ 2 & e \in \{(000, 001)\} \\ 3 & e \in \{(000, 100), (011, 001)\} \\ 4 & e \in \{(010, 011)\} \end{cases} \tag{1}$$



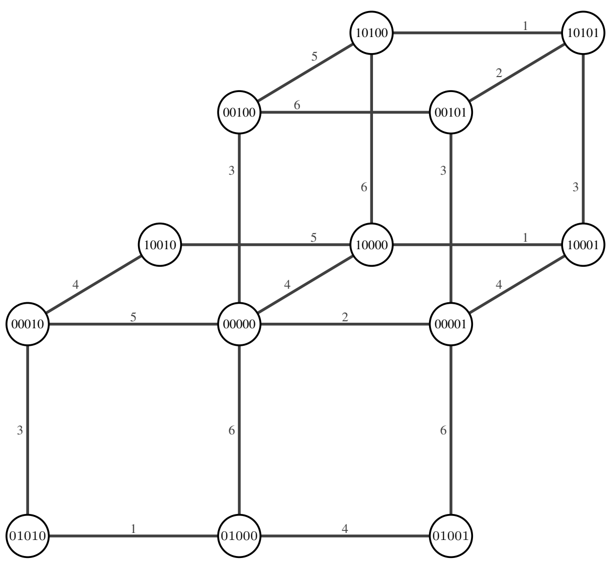
$n = 4$:

$$\phi(e) = \begin{cases} 1 & e \in \{(0000, 0100), (1000, 1010), (0001, 1001)\} \\ 2 & e \in \{(0000, 0001), (0010, 1010)\} \\ 3 & e \in \{(0000, 1000), (0100, 0101)\} \\ 4 & e \in \{(0001, 0101), (0000, 0010), (1000, 1001)\} \end{cases} \tag{2}$$



and $n = 5$:

$$\phi(e) = \begin{cases} 1 & e \in \{(10100,10101),(10000,10001), (01010,01000)\} \\ 2 & e \in \{(00101,10101), (00000,00001)\} \\ 3 & e \in \{(00100,00000),(00101,00001), (10101,10001), (00010,01010)\} \\ 4 & e \in \{(10010,00010),(00000,10000), (00001,10001), (01000,01001)\} \\ 5 & e \in \{(00010,00000),(10010,10000), (00100,10100)\} \\ 6 & e \in \{(00100,00101),(10100,10000), (00000,01000), (00001, 01001)\} \end{cases} \tag{3}$$

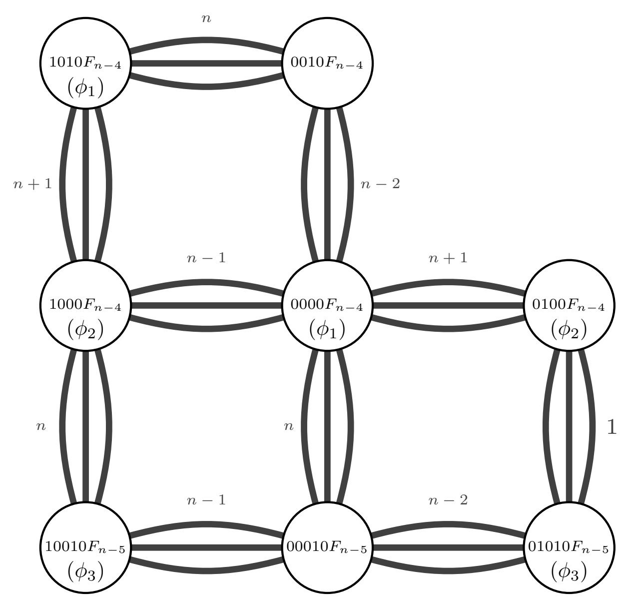


We will use these colorings for constructing coloring in general. Denote by $i_1 i_2 ... i_k F_{n-k}$ subgraph of F_n that contains vertices x with $i_1 i_2 ... i_k$ prefix, $i_l \in \{0, 1\}$ for $l = 1, ..., k$.

Suppose we have a vertex-distinguishing edge coloring for graph F_{n-4} that uses $n - 3$ colors and for graph F_{n-5} that uses $n - 4$ colors. We will split the graph into subgraphs and edges connecting adjacent vertices from different subgraphs. We will refer to the edges, that don't belong to the subgraphs as *connector edges*.

F_n can be represented as a combination of $1010F_{n-4}$, $1000F_{n-4}$, $10010F_{n-5}$, $0010F_{n-4}$, $0000F_{n-4}$, $00010F_{n-5}$, $0100F_{n-4}$, $01010F_{n-5}$ and $1000F_{n-4} - 1010F_{n-4}$, $1000F_{n-4} - 10010F_{n-5}$, $1010F_{n-4} - 0010F_{n-4}$, $1000F_{n-4} - 0000F_{n-4}$, $10010F_{n-5} - 00010F_{n-5}$, $0000F_{n-4} - 0010F_{n-4}$, $0000F_{n-4} - 00010F_{n-5}$, $0000F_{n-4} - 0100F_{n-4}$, $00010F_{n-5} - 01010F_{n-5}$, $0100F_{n-4} - 01010F_{n-5}$ connector edges.

For all those subgraphs $i_1 i_2 ... i_k F_{n-k}$, we use $n - k - coloring$ using colors $[1, n - k]$, except for subgraphs $0100F_{n-4}$ and $01010F_{n-5}$ for which we use colors $[2, n-3]$ and $[2, n - 4]$ correspondingly, which allows us to use 1 color for the connector edges between them. Thus, for subgraphs we use colors $[1, n - 3]$ and we are free to use $\{n-2, n-1, n, n+1\}$ colors for connector edges to construct a $n + 1 - coloring$.



We use the following coloring for the connector edges:

$$\begin{cases} n-2 & e \in \{0010F_{n-4} - 0000F_{n-4},\ 00010F_{n-5} - 01010F_{n-5}\} \\ n-1 & e \in \{1000F_{n-4} - 0000F_{n-4},\ 10010F_{n-5} - 00010F_{n-5}\} \\ n & e \in \{1010F_{n-4} - 0010F_{n-4},\ 1000F_{n-4} - 10010F_{n-5}\} \\ n+1 & e \in \{1010F_{n-4} - 1000F_{n-4},\ 0000F_{n-4} - 0100F_{n-4}\} \end{cases} \tag{4}$$

Let's now prove that the constructed coloring is vertex-distinguishing. For subgraphs vertices we use the following coloring of adjacent edges:

$$S(v) = \begin{cases} [1, n-4] \cup \{n, n+1\} & v \in 1010F_{n-4} \\ [1, n-4] \cup \{n-1, n, n+1\} & v \in 1010F_{n-4} \\ [1, n-5] \cup \{n-1, n\} & v \in 10010F_{n-5} \\ [1, n-4] \cup \{n-2, n\} & v \in 0010F_{n-4} \\ [1, n-4] \cup [n-2, n+1] & v \in 0000F_{n-4} \\ [1, n-5] \cup [n-2, n] & v \in 00010F_{n-5} \\ [1, n-4] \cup \{1, n+1\} & v \in 0100F_{n-4} \\ [1, n-5] \cup \{1, n-2\} & v \in 01010F_{n-5} \end{cases} \tag{5}$$

We colored edges of all the subgraphs using VDP-coloring by induction. It is easy to see that we used unique set of connector edge colors for all the subgraphs of the graph, meaning that for all the vertices of the graph different set of adjacent edge colors used. We constructed an $n+1-coloring$, which proves the upper bound of chromatic index for the graph. Note that for some fibonacci cubes (e.g. F_3) there is no VDP-coloring that uses n colors, meaning that the result can't be improved in general.

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