On Nonconvexity of the Set of Hypergraphic Sequences

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***Abstract*—In this paper, we prove that $D_m(n)$, the set of hypergraphic sequences of all simple hypergraphs $([n], E)$, where $[n] = \{1, 2, \cdots, n\}$, and $|E| = m$; being a subset of n-dimensional $m+1$-valued grid Ξ_{m+1}^n, is not a convex set in Ξ_{m+1}^n; also, we characterize the smallest convex set containing $D_m(n)$.**

***Keywords*—Hypergraphic sequences, non-convexity**

I. Introduction

The existence of simple uniform hypergraphs with a given degree sequence was a long-standing open problem ([1-6]); in 2018, the NP-completeness of the problem was proved [7]. The existence of simple hypergraphs with a given degree sequence (without given sizes of hyperedges) is not easier than the case of uniform hypergraphs ([8]). Characterization of $D_m(n)$, - the set of all degree sequences of simple hypergraphs with n vertices and m hyperedges, is investigated in [9-12]. The problem has its interpretation in terms of multidimensional binary cubes; it is also known as a special case in discrete tomography problems, when an additional constraint/requirement – non-repetition of rows, is imposed [13-14]. Structures, properties, and several related results were also obtained for $D_m(n)$. Convex hull of degree sequences of k-uniform hypergraphs was investigated in [4], [15-17]. In [16], it is verified computationally that the set of degree sequences for k-uniform hypergraphs is the intersection of a lattice and a convex polytope for $k = 3$ and $n \leq 8$. [17] shows that this does not hold for $k \geq 3$and $n \geq k + 13$.

In this paper, we prove that $D_m(n)$, being a subset of the n-dimensional $m + 1$-valued grid Ξ_{m+1}^n, is not a convex set in Ξ_{m+1}^n; also, we characterize the smallest convex set containing $D_m(n)$. This paper is an extended version of [18], where some preliminary results were presented without proofs.

The rest of the paper is organized as follows. Section 2 presents necessary definitions, preliminaries, and basic concepts. Main results are given in Section 3.

II. Preliminaries

A. Hypergraph degree sequences

A *hypergraph* H is a pair (V, E), where V is the vertex set of H, and E, the set of hyperedges, is a collection of non-empty subsets of V. The degree of a vertex v of H, denoted by $d(v)$, is the number of hyperedges in H containing v. A hypergraph H is simple if it has no repeated hyperedges. A hypergraph H is r-uniform if all hyperedges contain r-vertices.

Let $V = \{v_1, v_2, \cdots, v_n\}$. $D(H) = (d(v_1), d(v_2), \cdots, d(v_n))$ is the degree sequence of hypergraph H. A sequence $d = (d_1, d_2, \cdots, d_n)$ is hypergraphic if there is a simple hypergraph H with the degree sequence d. For a given m, $0 < m \leq 2^n$, let $H_m(n)$ denote the set of all simple hypergraphs $([n], E)$, where $[n] = \{1,2, \cdots, n\}$, and $|E| = m$; and $D_m(n)$ denote the set of all hypergraphic sequences of hypergraphs in $H_m(n)$.

B. Monotone Boolean functions

Let $B^n = \{(x_1, \cdots, x_n) \mid x_i \in \{0,1\}, i = 1, \cdots, n\}$ denote the set of vertices of the n-dimensional binary (unit) cube.

We define also *partition/splitting* of B^n into two $(n-1)$-dimensional sub-cubes according to the values of the binary variables; for arbitrary x_i:

$$B_{x_i=0}^{n-1} = \{(x_1, \cdots, x_n) \in B^n | x_i = 0\} \text{ and}$$
$$B_{x_i=1}^{n-1} = \{(x_1, \cdots, x_n) \in B^n | x_i = 1\}.$$

Any subset $\mathcal{M} \subseteq B^n$ will be partitioned into

$$\mathcal{M}_{x_i=1} \subseteq B_{x_i=1}^{n-1} \text{ and } \mathcal{M}_{x_i=0} \subseteq E_{x_i=0}^{n-1}.$$

An integer vector $S = (s_1, \cdots, s_n)$ is called *associated vector of partitions* of the set $\mathcal{M} \subseteq E^n$, if $s_i = |\mathcal{M}_{x_i=1}|$, $i = 1, \cdots, n$.

Boolean function $f: B^n \to \{0,1\}$ is called *monotone* if for every two vertices $\alpha, \beta \in B^n$, if $\alpha \prec \beta$ then $f(\alpha) \leq f(\beta)$. Vertices of B^n, where f takes the value "1" are called *units* or *true points* of the function; vertices, where f takes the value "0" are called *zeros* or *false points* of the function.

C. Characterization of $D_m(n)$

Clearly, every integer sequence of length n with all component values between 0 and m, can serve potentially as

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a degree sequence of some hypergraph with the vertex set $[n]$ and with m hyperedges. Thus, $D_m(n) \subseteq \{(a_1, \cdots, a_n) | 0 \leq a_i \leq m\}$; we denote this set by Ξ_{m+1}^n. We place a component-wise partial order on Ξ_{m+1}^n: $(a_1, \cdots, a_n) \preccurlyeq (b_1, \cdots, b_n)$ if and only if $a_i \leq b_i$ for all i. $(\Xi_{m+1}^n, \preccurlyeq)$ is a partial ordered set for which the rank of an element is given by $r(a_1, \cdots, a_n) = a_1 + \cdots + a_n$.

<u>Opposite elements in Ξ_{m+1}^n</u>

A pair of elements $(d, \bar{d})$ of Ξ_{m+1}^n are called opposite if one can be obtained from the other by inversions of component values, i.e., if $d = (d_1, \cdots, d_n)$, then $\bar{d} = (m - d_1, \cdots, m - d_n)$.

<u>Boundary elements of $D_m(n)$</u>

$(d_1, \cdots, d_n) \in D_m(n)$ is an *upper boundary /lower boundary/ element* of $D_m(n)$ if no $(a_1, \cdots, a_n) \in \Xi_{m+1}^n$ with $(a_1, \cdots, a_n) > (d_1, \cdots, d_n)$ / with $(a_1, \cdots, a_n) < (d_1, \cdots, d_n)$ / belongs to $D_m(n)$.

Let $\widehat{D}_{max}$ and $\breve{D}_{min}$ denote the sets of upper and lower boundary elements of $D_m(n)$, respectively.

<u>Interval/subgrid in Ξ_{m+1}^n.</u>

For a pair of elements d', d'', of Ξ_{m+1}^n with $d' \leq d''$, $E(d', d'')$ denotes the minimal subgrid/interval in Ξ_{m+1}^n spanned by these elements, i.e., $E(d', d'') = \{a \in \Xi_{m+1}^n | d' \leq a \leq d''\}$.

We will need also some preliminary results from [Sah, 2009]:

Lemma 1. $d = (d_1, \cdots, d_i, \cdots, d_n)$ belongs to $D_m(n)$ if and only if $\bar{d}_i = (d_1, \cdots, m - d_i, \cdots, d_n)$ belongs to $D_m(n)$, for arbitrary $i, 1 \leq i \leq n$.

Lemma 2. For each element $\hat{d} \in \widehat{D}_{max}$ there exists its opposite element $\bar{\hat{d}} \in \breve{D}_{min}$, and vice versa. Thus, $|\widehat{D}_{max}| = |\breve{D}_{min}|$.

Lemma 3.

For every element $\hat{d} = (\hat{d}_1, \cdots, \hat{d}_n)$ of $\widehat{D}_{max}$ $\hat{d}_i \geq m - \hat{d}_i$; and for every element $\check{d} = (\check{d}_1, \cdots, \check{d}_n)$ of $\breve{D}_{min}$ $\check{d}_i \leq m - \check{d}_i, i = 1, \cdots, n$.

Let d_{min} denote the element of $\widehat{D}_{max}$, which has the minimum rank among all elements of $\widehat{D}_{max}$, $r(d_{min}) = \min_{d \in \widehat{D}_{max}} r(d)$.

Lemma 4.

d_{min} has components equal to m, if $m \leq 2^{n-1}$.

Theorem 1. $D_m(n) = \bigcup_{\widehat{D} \in \widehat{D}_{max}, \breve{D} \in \breve{D}_{min}} E(\breve{D}, \widehat{D})$, where $(\widehat{D}, \breve{D})$ are pairs of opposite elements.

It is worth noting the relation of $\widehat{D}_{max}$ to the monotone Boolean functions defined on B^n. Each subset of vertices of B^n can be identified with the set of units of some Boolean function. In this manner, monotone Boolean functions represent a specific class of sets in B^n. Let M_m denote the class of m-sets in B^n represented by monotone Boolean functions with m units, and let $D_{M_m}(n)$ denote the class of corresponding associated vectors of partitions.

Theorem 2.

$\widehat{D}_{max} \subseteq D_{M_m}(n)$.

III. NON-CONVEXITY OF $D_m(n)$ IN Ξ_{m+1}^n

Ξ_{m+1}^n is an n-dimensional integral polytope, - a convex polytope the vertices of which have all integer coordinates between 0 to m. Undefined terms can be found in [19-20].

By definition, the intervals $E(\breve{D}, \widehat{D})$ are convex subsets in Ξ_{m+1}^n.

In this section, we prove that $D_m(n)$, being a union of convex sets $E(\breve{D}, \widehat{D})$, is not convex in Ξ_{m+1}^n.

Theorem 3. $D_m(n)$ is convex for $m = 1$, $2^n - 1$, 2^n, and not convex for $1 < m < 2^n - 1$.

Proof.

a) $m = 1$

There exists a unique monotone Boolean function with the single unit vertex $(1,1, \cdots, 1)$ of B^n. Therefore, $\widehat{D}_{max}$ consists of the single element $(m, m, \cdots, m)$, and this is the only possible case that $\widehat{D}_{max}$ contains $(m, m, \cdots, m)$. According to Lemma 2, $\breve{D}_{min}$ contains the single element $(0,0, \cdots, 0)$. Then, $D_m(n) = E((0,0, \cdots, 0), (m, m, \cdots, m))$, and this coincides with Ξ_{m+1}^n.

b) $m = 2^n$

There exists a unique monotone Boolean function, with the set of unit vertices coinciding with the whole B^n.

c) $m = 2^n - 1$

There exists a unique monotone Boolean function, the set of unit vertices of which coincides with $B^n \backslash \{(0,0, \cdots, 0)\}$.

Thus, in b) and c), $\widehat{D}_{max}$ consists of a single element with components equal to 2^{n-1}, and this is the only possible case that $\widehat{D}_{max}$ contains such an element. Hence, $D_m(n) = E((2^{n-1}, \cdots, 2^{n-1}), (2^{n-1}, \cdots, 2^{n-1}))$.

Thus, in a)-c), $D_m(n)$ is convex.

d) $1 < m < 2^n - 1$.

Let $\widehat{D}_{max} = \{\widehat{D}_1, \cdots, \widehat{D}_r\}$, $\breve{D}_{min} = \{\breve{D}_1, \cdots, \breve{D}_r\}$; $\widehat{D}_i$, $\breve{D}_i$ are opposite elements.

We prove that there exist $\breve{D}_i \in \breve{D}_{min}$ and $\widehat{D}_j \in \widehat{D}_{max}$, $i \neq j$ such that $E(\breve{D}_i, \widehat{D}_j)$ is not contained in $D_m(n)$.

Firstly, we notice that $\breve{D}_i \leq \widehat{D}_j$ for arbitrary i, j, since the components' values of $\widehat{D}_j$ are greater or equal to the middle value $\lceil m/2 \rceil$, and the components' values of $\breve{D}_i$- are less than or equal to the middle value $\lfloor m/2 \rfloor$ (according to Lemma 3).

Consider the following cases:

1) $m \leq 2^{n-1}$.

Let $\widehat{D}_j$ be a minimal element of $\widehat{D}_{max}$ (assume that components are in decreasing order): $\widehat{D}_j = (m, \hat{d}_2^j, \cdots, \hat{d}_n^j)$ (according to Lemma 4, it has m valued component). Consider another element $\widehat{D}_i = (\hat{d}_1^i, \hat{d}_2^i, \cdots, \hat{d}_n^i)$ of $\widehat{D}_{max}$, where $\hat{d}_1^i < m$. Such an element exists – it can simply be the vector obtained from $\widehat{D}_j$ by components permutation, taking into account also that all the components of $\widehat{D}_j$ cannot be equal to m.

Consider the opposite to $\widehat{D}_i$ element: $\breve{D}_i = (m - \hat{d}_1^i, m - \hat{d}_2^i, \cdots, m - \hat{d}_n^i)$, and replace the first component with m; we obtain $(m, m - \hat{d}_2^i, \cdots, m - \hat{d}_n^i)$, which belongs to $E(\breve{D}_i, \widehat{D}_j)$, but does not belong to $D_m(n)$, since according to Lemma 1,

$(m, \hat{d}_2^i, \cdots, \hat{d}_n^i)$ should belong to $D_m(n)$, which contradicts the fact that $\widehat{D}_i$ is an element of $\widehat{D}_{max}$.

2) $m > 2^{n-1}$.

The proof is similar to the previous case, taking into account that all components of $\widehat{D}_{max}$ cannot be equal to 2^{n-1}, besides the case of $m = 2^n - 1$. □

As an example, consider $D_4(3)$ in Ξ_5^3 given in Fig.1. (0,2,2) and (3,3,3) belong to $D_4(3)$, and $(0,2,2) < (3,3,3)$. However, the elements (0,3,2), (0,2,3), (0,3,3) of $\Xi 3_5^n$, which are greater than (0,2,2), and less than (3,3,3), - do not belong to $D_4(3)$.

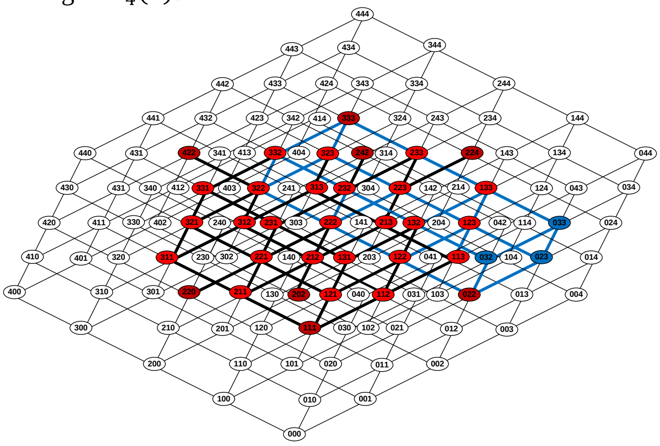

Fig. 1. Nonconvexity example

IV. The smallest convex set containing $D_m(n)$

In this section, we characterize the smallest convex subset of Ξ_{m+1}^n, containing $D_m(n)$. We denote this set by $C_{D_m(n)}$.

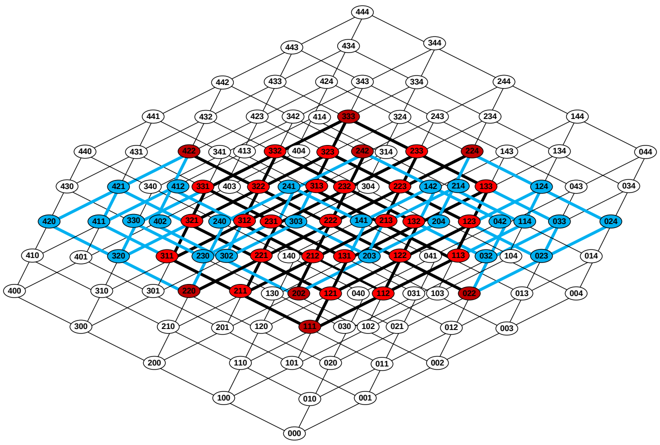

Fig. 2. Elements of $C_{D_4(3)}$ are colored (red and blue); elements of $D_4(3)$ are in red color.

Theorem 4. $C_{D_m(n)} = \bigcup_{i=1}^r \bigcup_{j=1}^r E(\breve{D}_i, \widehat{D}_j)$.

Proof.

It is clear that $D_m(n) \subseteq \bigcup_{i=1}^r \bigcup_{j=1}^r E(\breve{D}_i, \widehat{D}_j)$. Now we prove that $\bigcup_{i=1}^r \bigcup_{j=1}^r E(\breve{D}_i, \widehat{D}_j)$ is a convex set in Ξ_{m+1}^n, and there is no smaller set in Ξ_{m+1}^n, that contains $D_m(n)$.

Firstly, we prove that $\bigcup_{i=1}^r \bigcup_{j=1}^r E(\breve{D}_i, \widehat{D}_j)$ is convex in Ξ_{m+1}^n. Let $a, b \in \bigcup_{i=1}^r \bigcup_{j=1}^r E(\breve{D}_i, \widehat{D}_j)$, and $a < b$; we prove that the interval $[a, b] = \{c \in \Xi_{m+1}^n | a \leq c \leq b\}$ belongs to $\bigcup_{i=1}^r \bigcup_{j=1}^r E(\breve{D}_i, \widehat{D}_j)$, as well. If a, b are boundary elements (upper or lower), or belong to some $E(\breve{D}_i, \widehat{D}_i)$, then the proof is evident. Suppose that a, b are not boundary elements, and $a \in E(\breve{D}_i, \widehat{D}_i)$, $b \in E(\breve{D}_j, \widehat{D}_j)$, $i \neq j$. In this case, every element c from $[a, b]$ belongs to $E(\breve{D}_i, \widehat{D}_j)$ /taking into account that $\breve{D}_i \leq \widehat{D}_j$, for arbitrary $1 \leq i, j \leq r$/.

On the other hand, $\bigcup_{i=1}^r \bigcup_{j=1}^r E(\breve{D}_i, \widehat{D}_j) \subseteq C_{D_m(n)}$, - which implies that there is no smaller set in Ξ_{m+1}^n, that contains $D_m(n)$. □

Fig.2 demonstrates $C_{D_4(3)}$ in Ξ_5^3.

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