Maximal k-Sum-Free Collections in an Abelian Group

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***Abstract*—Let G be an Abelian group of order n, let $k \geqslant 2$ be an integer, and $A_1, \dots, A_k$ be non-empty subsets of G. The collection $(A_1, \dots, A_k)$ is called k-sum-free (abbreviated k-SFC), if the equation $x_1 + \dots + x_k = 0$ has no solutions in the collection $(A_1, \dots, A_k)$, where $x_1 \in A_1, \dots, x_k \in A_k$. The family of k-SFC in G will be denoted by $SFC_k(G)$. The collection $(A_1, \dots, A_k) \in SFC_k(G)$ is called maximal by capacity if it is maximal by the sum of $|A_i| + \dots + |A_k|$, and maximal by inclusion if for any $i \in \{1, \dots, k\}$ and $x \in G \setminus A_i$, the collection $(A_1, \dots, A_{i-1}, A_i \cup \{x\}, A_{i+1}, \dots, A_k) \notin SFC_k(G)$. Suppose $\varrho_k(G) = |A_i| + \dots + |A_k|$. In this work, we study the problem of the maximal value of $\varrho_k(G)$. In particular, the maximal value of $\varrho_k(Z_d)$ for the cyclic group Z_d is determined. Upper and lower bounds for $\varrho_k(G)$ are obtained for the Abelian group G. The structure of maximal k-sum-free set by capacity (by inclusion) is described for an arbitrary cyclic group.**

***Keywords*— Collection, sum-free, cyclic group, non-trivial subgroup, canonical homomorphism, coset.**

I. Introduction

Let G be an Abelian group of order n, let $k \geqslant 2$ be an integer, and $A_1, \dots, A_k$ be non-empty subsets of G. The collection $(A_1, \dots, A_k))$ is called a k-sum-free (abbreviated k-SFC) if there is no such a collection as

$$(a_1, \dots, a_k) \in A_1 \times \dots \times A_k,$$

being the solution of the equation

$$x_1 + \dots + x_k = 0. \qquad (1)$$

The family of k-SFC in G will be denoted by $SFC_k(G)$. Suppose

$$\varrho_k(G) = \max_{(A_1, \dots, A_k) \in SFC_k(G)} |A_1| + \dots + |A_k|.$$

Let $(A_1, \dots, A_k)$ be a k-sum-free collection in the group G. The collection $(A_1, \dots, A_k)$ is called maximal by capacity if it is maximal by $\varrho_k(G)$, and maximal by inclusion if for any $i \in \{1, \dots, k\}$ and $x \in G \setminus A_i$, the collection

$$(A_1, \dots, A_{i-1}, A_i \cup \{x\}, A_{i+1}, \dots, A_k)$$

is not k-sum-free in the group G.

In this article, the following issues are considered.

Problem 1. Definition of $\varrho_k(G)$.

Problem 2. The structure definition of maximal k-SFC by capacity (by inclusion).

II. Definition and Auxiliary Statements

Let $A_1, ..., A_k$ be non-empty subsets of the group G. Suppose

$$A_1 + \dots + A_k = \{x_1 + \dots + x_k \mid x_1 \in A_1, \dots, x_k \in A_k\}.$$

If $(A_1, \dots, A_k) \in SFC_k(G)$, then this is equivalent to the fact that $0 \notin A_1 + \dots + A_k$.

Let G be an Abelian group and let H be a subgroup of G. Then, through $\phi_{G,G/H}$ the ultimate canonical homomorphism $\phi_{G,G/H} : G \to G/H$, and for any subset A of the group G, we denote by A/H the subset $\phi_{G,G/H}(A)$ of the group factor G/H.

Lemma 1: Let H be a subgroup of the Abelian group G. If $(A_1, \dots, A_k) \in SFC_k(G/H)$, then $(\phi^{-1}_{G,G/H}(A_1), \dots, \phi^{-1}_{G,G/H}(A_k)) \in SFC_k(G)$. Moreover, if $(\phi^{-1}_{G,G/H}(A_1), \dots, \phi^{-1}_{G,G/H}(A_k))$ is a maximal (by inclusion) k-sum-free collection in the group G, then $((A_1, \dots, A_k)$ is a maximal (by inclusion) k-sum-free collection in the group factor G/H.

Definition 1: Let A be a non-empty subset of the group G. The largest subgroup $H(A)$ of the group G such that $A + H(A) = A$, is called a stabilizer of the set A.

Let G be an Abelian group, let $k \geqslant 2$ be an integer, and $A_1, \dots, A_k$ be non-empty subsets of G. Here $H_G = H_G(A_1 + \dots + A_k)$ denotes the stabilizer of the set $A_1 + \dots + A_k$

$$A_1 + \dots + A_k + H_G = A_1 + \dots + A_k. \qquad (2)$$

Lemma 2: If $(A_1, \dots, A_k)$ is a maximal k-sum-free collection by inclusion in the group G, then the set $(A_1 + H_G, \dots, A_k + H_G)$ is also a maximal k-sum-free set by inclusion in the group G.

Lemma 3: If $(A_1, \dots, A_k)$ is a maximal k-sum-free collection by inclusion in the group G, then for any $i \in \{1, \dots, k\}$ the collection $(A_1, \dots, A_{i-1}, A_i + H_G, A_{i+1}, \dots, A_k)$ is also a maximal k-sum-free collection by inclusion in the group G.

Lemma 4: If $(A_1, \dots, A_k)$ is a maximal k-sum-free collection by inclusion in the group G, then $A_i = A_i + H_G$, and hence, A_i represents a combination of several adjacent classes of the subgroup H_G, which in turn means that $|A_i|$ is divided into $|H_G|$, for all $i = 1, \dots, k$.

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Lemma 5: If $(A_1, \dots, A_k)$ is a maximal k-sum-free collection by inclusion in the group G, then the set $(A_1/H_G, \dots, A_k/H_G)$ is also a maximal k-sum-free collection by inclusion in the group G/H_G.

Lemma 6: Let G be an Abelian group, let $(A_1, \dots, A_k)$ be a maximal k-sum-free collection by inclusion in the group G, and H_G be a stabilizer of the set $A_1 + \dots + A_k$, and H_{G/H_G} be a stabilizer of the set $A_1/H_G + \dots + A_k/H_G$. Then $H_{G/H_G} = H_G/H_G = \{0\} \in G/H_G$.

Lemma 7: If $(A_1, \dots, A_k)$ is a k-sum-free collection in the Abelian group G, then for any $2 \leqslant m \leqslant k-1$ $(A_1, \dots, A_{m-1}, A_m + \dots + A_k)$ it is an m-sum-free collection in the Abelian group G.

III. Definition of $\varrho_k(G)$

In 1813, Cauchy [1] proved the first result of a theory called *Additive number theory*. The Additive number theory is the main tool for studying **Problem 1** and **Problem 2**. Cauchy's result, which Davenport [2], [3] revised in 1935, is called the Cauchy–Davenport theorem. Applying this theorem we get the exact value $\varrho_k(Z_p)$ for a cyclic group of a simple order.

Theorem 1: For any prime number p the following equality is true

$$\varrho_k(Z_p) = p + k - 2.$$

In 1953, Kneser [4], [5] generalized Cauchy–Davenport's result for any Abelian group. Applying Kneser's theorem we obtain lower and upper bounds for any Abelian groups.

Theorem 2: Let G be an Abelian group of order n and exponent ν. Then

$$n + \frac{n}{p_1}(k-2) = \max_{d|\nu}\left(\frac{n}{d}(d+k-2)\right) \leqslant$$
$$\leqslant \varrho_k(G) \leqslant$$
$$\leqslant \max_{d|n}\left(\frac{n}{d}(d+k-2)\right) = n + \frac{n}{p_2}(k-2),$$

where p_1 is the smallest prime divisor of ν, and p_2 is the smallest prime divisor of n.

There exist premises to imply that the following statement is true.

Theorem 3 (Hypothesis): Let G be an Abelian group of order n and exponent ν. Then

$$\varrho_k(G) = n + \frac{n}{p}(k-2),$$

where p is the smallest prime divisor of ν.

Theorem 4: For any n, the following equality is true:

$$\varrho_k(Z_n) = n + \frac{n}{p}(k-2),$$

where p is the smallest prime divisor of n.

Theorem 5: Let G be an Abelian group of order n and exponent ν. Then

$$\varrho_k(G) \geq \max_{d|\nu}\left(\frac{n}{d}\varrho_k(Z_d)\right).$$

IV. On the Structure of a Maximal by Capacity k-Sum-Free Collection in a Cyclic Group

Let A be a subset of the Abelian group G, then denote by $\overline{A}$ as a complement of the subset A in the Abelian group G, that is, $\overline{A} = G \setminus A$, and for any natural number m denote $m\star A = \{ma \mid a \in A\}$ and$m\star A$ will be called an extension of the set A. Let's define $\mathrm{ord}(A) = \{\mathrm{ord}(a) \mid a \in A\}$, where $\mathrm{ord}(a)$ is the order of the element a.

Lemma 8: Let $(A_1, \dots, A_k)$ be a k-sum-free collection in the Abelian group G. Then for any $m \notin \mathrm{ord}(A_1 + \dots + A_k)$ the collection $(m \star A_1, \dots, m \star A_k)$ is k-sum-free in the Abelian group G.

Remark 1: Note that for any $A \subseteq Z_p$, where p is a prime number, $\mathrm{ord}(A) = \{p\}$.

Definition 2: The arithmetic progression of P in the Abelian group G is such an entity that there exist two elements $a, d \in G$, and a non-negative integer s, such that

$$P = \{a + jd \mid 1 \leqslant j \leqslant s\}.$$

In 1956, Vosper [6], [7] considered the Cauchy–Davenport result in the case of equality. Vosper's theorem will mainly help in the study of **Problem 2**.

As a result, we got the following result:

Lemma 9: Let $(A_1, \dots, A_k)$ be a k-sum-free collection in the cyclic group of the prime order Z_p such that the difference of all arithmetic progressions in $\{A_1, \dots, A_k\}$ is equal to d. Let $dm(mod\, p) = 1$. Then $(m \star A_1, \dots, m \star A_k)$ is a k-sum-free collection in Z_p, and the difference of all arithmetic progressions in $\{m \star A_1, \dots, m \star A_k\}$ is equal to 1.

Lemma 10: If $(A_1, \dots, A_k)$ is a maximal k-sum-free collection by capacity in the cyclic group of prime order Z_p, then for any $2 \leqslant m \leqslant k-1$ $(A_1, \dots, A_{m-1}, A_m + \dots + A_k)$ it is a maximal m-sum-free collection by capacity in Z_p.

Theorem 6: Let $k \geqslant 2$, and $A_1, \dots, A_k$ be non-empty subsets of the cyclic group Z_p of the prime order p, such that $A_1 + \dots + A_k \neq Z_p$. Then $|A_1 + \dots + A_k| = |A_1| + \dots + |A_k| - (k-1)$, if and only if for each set $A_{k-i}, i = 0, \dots, k-1$, there occurs at least one of the following three conditions:

(i) $\min(|A_1 + \dots + A_{k-i-1}|, |A_{k-i}|) = 1$;
(ii) if $|A_1 + \dots + A_{k-i}| = p - 1$, then $A_{k-i} = \overline{c - (A_1 + \dots + A_{k-i-1})}$, where $\{c\} = \overline{(A_1 + \dots + A_{k-i})}$;
(iii) $A_1 + \dots + A_{k-i-1}, A_{k-i}$ are arithmetic progressions with the same difference.

Remark 2: Since the permutation keeps the collection sum-free, that is, if the collection $(A_1, \dots, A_k)$ is k-sum-free then the collection $(A_{i_1}, \dots, A_{i_k})$ is also k-sum-free where $(i_1, \dots, i_k)$ is an arbitrary permutation of the set $(1, \dots, k)$, then the sequence of choice of sets can be arbitrary.

Remark 3: All arithmetic progressions in $(A_1, \dots, A_k)$ have the same difference.

The next theorem describes the structure of each maximal k-sum-free collection by capacity (with accuracy up to isomorphism) in the cyclic group of prime order.

Theorem 7: Let $k \geqslant 2$, let Z_p be a cyclic group of prime order, and let $A_1, \dots, A_k$ be a maximal k-sum-free collection by capacity in Z_p. Then, each entity of the set with accuracy up to isomorphism is one of the following:

(i) $|A_i| = 1$;

(ii) $A_i = \overline{-(A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_k)}$;

(iii) A_i is an arithmetic progression with difference 1;

where $i = 1, \dots, k$.

Theorem 8: Let $k \geqslant 2$, and p be the smallest prime divisor of a natural number n, and H be a subgroup of the group Z_n of order n/p, and $(A_1, \dots, A_k)$ be a maximal k-sum-free collection by capacity in Z_n. Then, each entity of this set, with accuracy up to isomorphism, is one of the following:

(i) $|A_i| = n/p$, that is, A_i is the coset of Z_n by the subgroup H;

(ii) A_i is a union of cosets Z_n by the subgroup H such that for sets of representatives of cosets as subsets of the cyclic group Z_p, the following relation is correct: $A_i/H = \overline{-(A_1/H + .. + A_{i-1}/H + A_{i+1}/H + .. + A_k/H)}$;

(iii) A_i is a union of cosets Z_n by the subgroup H such that the set of representatives of cosets as a subset of the cyclic group Z_p, is an arithmetic progression with difference 1;

where $i = 1, \dots, k$.

It is well known that any finite Abelian group is isomorphic to some group of the form

$$Z/a_1Z \times \cdots \times Z/a_sZ,$$

where $2 \leqslant a_s | a_{s-1} | \dots | a_2 | a_1$ (see in [8]).

The following result is on one construction of the maximal by inclusion k-sum-free collection in a cyclic group.

Lemma 11: If in an Abelian group G there exists a maximal by inclusion k-sum-free collection with the capacity of k, then the group G is cyclic.

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