Strong Edge-Coloring of Hamming Graphs

Aram Drambyan
Russian-Armenian University
Yerevan, Armenia
e-mail: ardrambyan@student.rau.am

Petros Petrosyan
Yerevan State University
Yerevan, Armeina
e-mail: petros_petrosyan@ysu.am

***Abstract*—An edge-coloring ϕ of a graph G is called strong if any two edges at distance at most 2 receive different colors. The minimum number of colors required for a strong edge-coloring of a graph G is called a strong chromatic index of graph G and denoted by $\chi'_s(G)$. Hamming graph $H(n,m)$ is the Cartesian product of n complete graphs K_m. In this paper, for Hamming graphs $H(n,m)$, we show that $nm(m-1) - \frac{m(m-1)}{2} \leq \chi'_s(H(n,m)) \leq nm(m-1)$ if m is even and $nm(m-1) - \frac{m(m-1)}{2} \leq \chi'_s(H(n,m)) \leq nm^2$ if m is odd.**

***Keywords*— Edge Colorings, Strong edge colorings, Hamming Graphs.**

I. Introduction

All graphs considered in this paper are finite and simple. We denote by $V(G)$ and $E(G)$ the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d(v)$ and the maximum degree of vertices in G by $\Delta(G)$.

An *edge-coloring* of a graph G is a mapping $\phi : E(G) \to \mathbb{N}$. ϕ is called *strong* if any two edges at distance at most 2 receive different colors. The minimum number of colors required for a strong edge-coloring of a graph G is called a *strong chromatic index* of graph G and denoted by $\chi'_s(G)$. Strong edge-coloring of graphs was introduced by Fouquet and Jolivet in 1983 [1]. Later, during seminar in Prague, Erdős and Nešetřil proposed the following conjecture.

Conjecture 1. *For every graph G with maximum degree $\Delta(G)$,*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta(G)^2, & \text{if } \Delta(G) \text{ is even,} \\ \frac{1}{4}(5\Delta(G)^2 - 2\Delta(G) + 1), & \text{if } \Delta(G) \text{ is odd.} \end{cases}$$

Conjecture was proved for $\Delta(G) = 3$ by [2] and [3] independently. For $\Delta(G) = 4$, currently known best result is $\chi'_s(G) \leq 21$, which was proven by Huang et al. [6]. Also, Hurley, de Joannis de Verclos, and Kang [5] showed that $\chi'_s(G) \leq 1.772\Delta(G)^2$ for any graph with sufficiently large maximum degree $\Delta(G)$. This improves the old, well-known result of $\chi'_s(G) \leq 1.998\Delta(G)^2$ proved by Molloy and Reed [7].

Graph, where each pair of vertices are connected with an edge, is called *complete* and denoted by K_n. The *Cartesian product* $G\square H$ of graphs G and H is a graph with a set of vertices $V(G) \times V(H)$, and 2 vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $u_1 = v_1$ and u_2 and v_2 are adjacent in H or $u_2 = v_2$ and u_1 and v_1 are adjacent in G. *Hamming graph* $H(n,m)$ is the Cartesian product of n complete graphs K_m. *The hypercube, or n-cube* is a graph, the vertices of which can be represented as binary strings of length n, and two vertices u, v are adjacent if and only if their string representations are equal in all but one position and is denoted by Q_n. In 1990, Faudree, Schelp, Gyárfás and Tuza [4] showed that $\chi'_s(Q_n) = 2n$. It's easy to see that $H(n,2)$ is a hypercube, and the upper bound from this paper matches with the proven result for Q_n.

II. Main Result

We begin our considerations with the lower bound for strong chromatic index of Hamming graphs.

Theorem 1: Let $H(n,m)$ be a Hamming graph with $m \geq 2$. Then

$$\chi'_s(H(n,m)) \geq nm(m-1) - \frac{m(m-1)}{2}$$

Proof: Vertices of the Hamming graph can be represented as a tuples of length n, where each position can take a value from 0 to $m-1$, and 2 vertices are adjacent if and only if they are equal in all but one position. Let us consider m vertices $v_1 = (0,0,...,0)$, $v_2 = (1,0,...,0)$, ..., $v_m = (m-1,0,...,0)$. They all are at distance 1 from each other and all the edges, adjacent to that vertices, should receive different colors. For any vertex v from $H(n,m)$, $d(v) = n(m-1)$. We get $\chi'_s(H(n,m)) \geq md(v) - \dfrac{m(m-1)}{2} = nm(m-1) - \dfrac{m(m-1)}{2}$.□

We continue with upper bound for strong chromatic index of Hamming graphs without proof.

Theorem 2: Let $H(n,m)$ be a Hamming graph with $m \geq 2$. Then

$$\chi'_s(H(n,m)) \leq \begin{cases} nm(m-1), & \text{if } m \text{ is even,} \\ nm^2, & \text{if } m \text{ is odd.} \end{cases}$$

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