An Upper Bound on the Edge-Chromatic Sum of Fibonacci Cubes

Hamlet Mikaelyan
Yerevan State University
Yerevan, Armenia
e-mail: hamletm2000@gmail.com

Petros Petrosyan
Yerevan State University
Yerevan, Armenia
e-mail: petros_petrosyan@ysu.am

***Abstract*—Fibonacci cube is an isometric subgraph of the n-dimensional cube. A proper edge-coloring of a graph G is a mapping $\alpha : E(G) \longrightarrow \mathbb{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges e and e' in G. The edge-chromatic sum of a graph G is the minimum sum of all colors in the graph among all its proper edge-colorings. This paper provides an upper bound on the edge-chromatic sum of Fibonacci cubes.**

***Keywords*— Edge-chromatic sum, Fibonacci cubes, sum edge-coloring.**

I. Introduction

Let $B = \{0, 1\}$ and for $n \geq 1$ set $\mathcal{B}_n = \{b_1b_2...b_n \mid b_i \in B,\ 1 \leq i \leq n\}$. The n-cube Q_n graph is the graph defined on the vertex set $\mathcal{B}_n$, vertices $b_1b_2...b_n$ and $b'_1b'_2...b'_n$ being adjacent if $b_i \neq b'_i$ holds for exactly one $i \in \{1, 2, ..., n\}$.

Fibonacci cubes are introduced as follows: for $n \geq 1$, let $\mathcal{F}_n = \{b_1b_2...b_n \in \mathcal{B}_n \mid b_i \cdot b_{i+1} = 0,\ 1 \leq i \leq n-1\}$. The Fibonacci cube Γ_n is the subgraph of Q_n induced by the vertex set $\mathcal{F}_n$ [1].

A proper vertex-coloring of a graph G is a mapping $\alpha : V(G) \rightarrow \mathbb{N}$ such that $\alpha(u) \neq \alpha(v)$ for every $uv \in E(G)$. In that case $\alpha(v)$ is called the color of the vertex v. The vertex-chromatic sum $\Sigma(G)$ of a graph G is the minimum sum of colors of all vertices among all proper vertex-colorings of G. The concept of vertex-chromatic sum was introduced by Kubicka [2] and Supowit [3]. The problem of finding the vertex-chromatic sum is shown to be NP-complete in general and polynomial time solvable for trees [4]. Jansen [5] gave a dynamic programming algorithm for partial k-trees. In papers [6], [7], [8], [9], [10], some approximation algorithms were given for various classes of graphs. Some bounds for the vertex-chromatic sum of a graph were given in [11].

A proper edge-coloring of a graph G is a mapping $\alpha : E(G) \rightarrow \mathbb{N}$ such that $\alpha(e) \neq \alpha(e')$ for every adjacent e and e'. In that case $\alpha(e)$ is called a color of the edge e. Similar to the vertex-chromatic sum of graphs, in [6], [12], and [13], edge-chromatic sum of graphs was introduced. Namely, the edge-chromatic sum $\Sigma'(G)$ of a graph G is the minimum sum of all colors in the graph among all its proper edge-colorings. In [6], Bar-Noy et al. proved that the problem of finding the edge-chromatic sum is NP-hard for multigraphs. Later, in [12], it was shown that the problem is NP-complete for bipartite graphs with maximum degree 3. Petrosyan and Kamalian [14] proved that the problem is NP-complete for even more specific class of graphs from the latter and found an $\frac{11}{8}$-approximation algorithm for r-regular graphs. In [15], Salavatipour proved that determining the edge-chromatic sum is NP-complete for r-regular graphs with $r \geq 3$. The problem can be solved in polynomial time for trees [12].

The terms and concepts that we do not define can be found in [16].

In the present paper, we obtain an upper bound on the edge-chromatic sum of Fibonacci cubes.

II. Main result

Theorem 1. For any $n \in \mathbb{N}$, we have

$$\Sigma'(\Gamma_n) \leq$$

$$\leq \left(\frac{5+3\sqrt{5}}{100}n^2 + \frac{11+9\sqrt{5}}{100}n + \frac{6\sqrt{5}}{125}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n +$$

$$+\left(\frac{5-3\sqrt{5}}{100}n^2 + \frac{11-9\sqrt{5}}{100}n - \frac{6\sqrt{5}}{125}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Proof. We will construct a corresponding proper edge-coloring α_n for each Γ_n by induction on n. We denote by $\Sigma'(\alpha_n)$ the sum of colors of all edges for the coloring α_n. From the relation $\Sigma'(\Gamma_n) \leq \Sigma'(\alpha_n)$, which implies from the definition of the edge-chromatic sum, we will derive the result.

It is easy to construct α_1 and α_2 separately, so they have, respectively, 1 and 3 as their sums.

Now let us construct the coloring α_n for $n \geq 3$ assuming that we have already constructed all α_k-s for $1 \leq k < n$. It is known that it is possible to decompose Γ_n into two subgraphs Γ_{n-1} and Γ_{n-2} in such a way that $V(\Gamma_n) = V(\Gamma_{n-2}) \cup V(\Gamma_{n-1})$ and $E(\Gamma_n) = E(\Gamma_{n-2}) \cup E(\Gamma_{n-1}) \cup M$, where M is a matching of $2|V(\Gamma_{n-2})|$ vertices [1]. Let us color the edges of the matching with the color 1. For the remaining edges let us use the corresponding colors in the colorings α_{n-2} and α_{n-1}, and color the edge e with $\alpha_{n-2}(e)+1$ if $e \in E(\Gamma_{n-2})$ and $\alpha_{n-1}(e)+1$ if $e \in E(\Gamma_{n-1})$.

Clearly, we constructed a proper edge-coloring. Moreover, $\Sigma'(\alpha_n) = |E(\Gamma_n)| + \Sigma'(\alpha_{n-1}) + \Sigma'(\alpha_{n-2})$. By [1], we have that $|E(\Gamma_n)| = \frac{nF_{n+1}+2(n+1)F_n}{5}$, where F_n is the n-th Fibonacci number.

https://doi.org/10.51408/csit2023_31

If we denote $\Sigma'(\alpha_n)$ by $a(n)$, then to obtain the required inequality, it is necessary to solve the 2nd order non-homogeneous recurrence relation $a(n) = a(n-1)+a(n-2)+\frac{nF_{n+1}+2(n+1)F_n}{5}$ with initial conditions $a(1) = 1$ and $a(2) = 3$. To do that, we represent $a(n)$ as the sum $a_h(n) + a_p(n)$, where $a_h(n)$ is the solution of the associated homogeneous recurrence relation, and $a_p(n)$ is the particular solution. The characteristic equation of the homogeneous relation is $\lambda^2-\lambda-1$, roots of which are $\lambda = \frac{1+\sqrt{5}}{2}$ and $\lambda = \frac{1-\sqrt{5}}{2}$. Therefore, $a_h(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$. As for the particular solution, considering the formula of the common term of the Fibonacci sequence, we get that $a_p(n)$ has the following form: $(An^2+Bn+C)\left(\frac{1+\sqrt{5}}{2}\right)^n+(Dn^2+En+F)\left(\frac{1-\sqrt{5}}{2}\right)^n$. Thus, we obtained the forms of $a_h(n)$ and $a_p(n)$, and to get the formula of $a(n)$ we need to put those results in the recurrence relation, and obtain the values of the coefficients using the initial conditions. □

The proper edge-colorings α_3 and α_4 are illustrated in Figure 1.

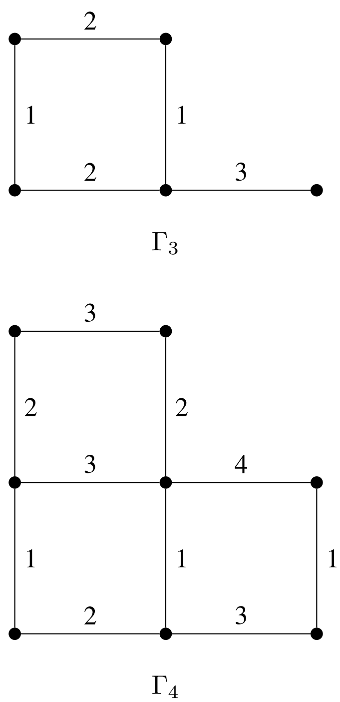

Figure 1

References

[1] S. Klavžar, "Structure of Fibonacci cubes: A survey", *Journal of Combinatorial Optimization 25*, pp. 505-522, 2013.

[2] E. Kubicka. *The chromatic sum of a graph*, PhD thesis, Western Michigan University, 1989.

[3] K.J. Supowit, "Finding a maximum planar subset of a set of nets in a channel", *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 6(1)*, pp. 93-94, 1987.

[4] E. Kubicka and A. J. Schwenk, "An introduction to chromatic sums", In *Proceedings of the 17th Conference on ACM Annual Computer Science Conference, CSC '89*, ACM Press, New York, pp. 39–45, 1989 .

[5] K. Jansen, "The optimum cost chromatic partition problem", In Giancarlo Bongiovanni, Daniel Pierre Bovet, and Giuseppe Di Battista, editors, *Algorithms and Complexity*, Springer Berlin Heidelberg, pp. 25-36, 1997.

[6] A. Bar-Noy, M. Bellare, M. Halldórsson, H. Shachnai and T. Tamir, "On chromatic sums and distributed resource allocation", *Information and Computation 140*, pp. 183-202, 1998.

[7] A. Bar-Noy and G. Kortsarz, "The minimum color sum of bipartite graphs, *J. Algorithms 28*, pp. 339-365, 1998.

[8] K. Giaro, R. Janczewski, M. Kubale and M. Małafiejski, "A 27/26-approximation algorithm for the chromatic sum coloring of bipartite graphs", *Lecture Notes in Computer Science 2462*, pp. 135-145, 2002.

[9] K. Jansen, "Approximation results for the optimum cost chromatic partition problem", *Journal of Algorithms 34*, pp. 54-89, 2000.

[10] E. Kubicka, G. Kubicki and D. Kountanis, "Approximation algorithms for the chromatic sum", In Naveed A. Sherwani, Elise de Doncker, and John A. Kapenga, editors, *Computing in the 90's*, pp. 15-21, 1991.

[11] C. Thomassen, P. Erdös, Y. Alavi, P.J. Malde and A.J. Schwenk, "Tight bounds on the chromatic sum of a connected graph", *Journal of Graph Theory 13*, pp. 353-357, 1989.

[12] K. Giaro and M. Kubale, "Edge-chromatic sum of trees and bounded cyclicity graphs", *Information Processing Letters 75*, pp. 65-69, 2000.

[13] H. Hajiabolhassan, M.L. Mehrabadi and R. Tusserkani, "Minimal coloring and strength of graphs", *Discrete Mathematics 215*, pp. 265-270, 2000.

[14] P.A. Petrosyan and R.R. Kamalian, "On sum edge-coloring of regular, bipartite and split graphs", *Discrete Appl. Math. 165*, pp. 263–269, 2014.

[15] M.R Salavatipour, "On sum coloring of graphs", *Discrete Appl. Math. 127*, pp. 477-488, 2003.

[16] D.B. West, *Introduction to Graph Theory*, Prentice Hall, New Jersey, 2001.