Robustness Analysis of Multivariable Uniform Control Systems

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***Abstract*—This paper presents simple graphical tests for analyzing the robustness of a special class of linear multi-input, multi-output (MIMO) feedback systems called uniform systems. The uniform systems are MIMO systems with identical transfer functions of separate channels and rigid cross-connections described by a square numerical matrix. The exposition is based on the method of characteristic transfer functions, which allows reducing the analysis of an interconnected MIMO system with N and N outputs to the analysis of N fictitious independent systems with one input and one output. The proposed robustness tests are in the form of N 'forbidden' circles on the complex plane of the characteristic gain loci of the open-loop uniform system. A numerical example illustrating the application of the tests to the analysis of the control system of a quadcopter is given.**

***Keywords* - Multivariable control system, uniform system, uncertainties, stability robustness, multirotor UAV**

I. INTRODUCTION

Robustness of multivariable control systems to disturbances and uncertainties has always been one of the central issues in feedback control [1]-[3]. The paper presents simple graphical tests for analyzing robustness to additive perturbations of a special class of linear multi-input and multi-output (MIMO) control systems called uniform systems. The uniform systems are MIMO systems with identical transfer functions of separate channels and rigid cross-connections described by a square numerical matrix. These specific structural features of uniform systems allow transforming the well-known sufficient conditions of robustness of MIMO systems to a very simple and visual form, which is very close to sufficient conditions of single-input, single-output (SISO) control systems. The exposition is based on the method of characteristic transfer functions (CTFs) [4], which allows reducing the stability analysis of an interconnected MIMO system with N and N outputs to the stability analysis of N fictitious independent SISO systems.

The proposed graphical tests of stability robustness of uniform systems to additive perturbations are very similar to the stability analysis of SISO control systems by the Nyquist criterion, in which the critical point $-1, j0$ is replaced by N 'forbidden' circles on the complex plane of the characteristic gain loci of the open-loop uniform system.

II. CANONICAL REPRESENTATION AND STABILITY ANALYSIS OF UNIFORM MIMO SYSTEMS

Matrix block diagram of a linear uniform MIMO system is shown in Fig. 1, where $w(s)$ is a scalar (SISO) transfer function of identical separate channels and R is an $N \times N$ numerical matrix of rigid cross-connections.

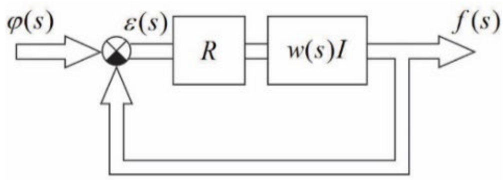

Fig. 1. Block diagram of a uniform MIMO system

The transfer matrix $W(s)$ of the open-loop uniform system in Fig.1:

$$W(s) = w(s)R \tag{1}$$

coincides, up to the complex scalar multiplier $w(s)$, with the numerical matrix of cross-connections R. The corresponding sensitivity and complementary sensitivity transfer matrices $S(s)$ and $T(s)$ of the closed-loop uniform system have the following standard forms [3]:

$$S(s) = [I + W(s)]^{-1}, \tag{2}$$

$$T(s) = [I + W(s)]^{-1} W(s). \tag{3}$$

Consider the canonical representations of the uniform MIMO system transfer matrices [4]. Denoting by λ_i the eigenvalues of R, which for simplicity are supposed distinct, and by C the modal matrix composed of linearly independent eigenvectors c_i of R, the canonical representation of the open-loop uniform system via similarity transformation will have the following form:

$$W(s) = C\, diag\{\lambda_i w(s)\}\, C^{-1}. \tag{4}$$

As can be seen from (1) and (4), the canonical basis of the linear uniform system is completely defined by the numerical matrix of cross-connections R and does not depend on the transfer function $w(s)$ of separate channels. Besides, all the CTFs

$$q_i(s) = \lambda_i w(s), \ (i = 1, 2, ..., N) \tag{5}$$

coincide, up to the constant "gains" λ_i, with the transfer function $w(s)$. Considering (4) and (5), the canonical representations of transfer matrices $S(s)$ (2) and $T(s)$ (2) are:

$$T(s) = C\,diag\left\{\frac{\lambda_i w(s)}{1 + \lambda_i w(s)}\right\} C^{-1}, \tag{6}$$

$$S(s) = C\,diag\left\{\frac{1}{1 + \lambda_i W(s)}\right\} C^{-1}. \tag{7}$$

The stability of the linear closed-loop uniform system is determined by the roots of the characteristic equation:

$$\det\left[1 + w(s)R\right] = \prod_{i=1}^{N}[1 + \lambda_i w(s)] = 0, \tag{8}$$

which is equivalent to a set of N equations:

$$1 + \lambda_i w(s) = 0, \quad (i = 1, 2, ..., N). \tag{9}$$

The fact that the CTFs $q_i(s)$ (4) differ from the transfer function $w(s)$ of separate channels only by the numerical coefficients λ_i simplifies the stability analysis of uniform systems based on the generalized Nyquist criterion. There are two possible formulations of that criterion called *direct* and *inverse* [4]. According to the "direct" formulation, if the transfer function $w(s)$ has k_0 poles in the right-half plane, then the closed-loop uniform MIMO system is stable if each characteristic gain locus $\lambda_i w(j\omega)$ encircles, as the frequency ω changes from $-\infty$ to $+\infty$, the critical point $(-1, j0)$ in the anticlockwise direction k_0 times.

In accordance with the second ('inverse') formulation, the equations (5) must be rewritten in the following form

$$w(s) = -1/\lambda_i, \quad i = 1, 2, ..., N. \tag{10}$$

In this case, a single graph $w(j\omega)$ and N critical points $-1/\lambda_i$ are plotted on the complex plane, and for stability of the closed-loop uniform system, it is necessary and sufficient that the Nyquist plot of $w(j\omega)$ does encircle each of N points $-1/\lambda_i$ in the anticlockwise direction k_0 times [4].

III. Basic Perturbation Models of Uniform Systems

In this section, we discuss some issues concerning the robustness of uniform systems. Nowadays, there are various paradigms for modeling dynamic system uncertainties, e.g., structured, unstructured, highly structured (or parametric), etc. [3].

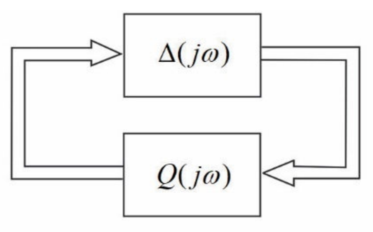

Fig. 2. Basic perturbation model of a MIMO control system

The most common approach to analyzing the influence of uncertainties on the stability of the system assumes that uncertainties may be represented in the form of the *Basic Perturbation Model* (BPM) shown in Fig. 2 [3]. Here, $Q(j\omega)$ is the transfer matrix of the ideal (nominal) system, which is assumed to be stable and the stability robustness of which is under investigation, and the block $\Delta(j\omega)$ represents all uncertainties of the dynamics of the system.

One of the key results in the robust theory is based on the *small gain theorem* [1,2], and is formulated for the systems in Fig. 2 as follows:

Let $Q(j\omega)$ and $\Delta(j\omega)$ be stable. Then, for stability of the MIMO system with uncertainty $\Delta(j\omega)$ it is sufficient that for all frequencies ω, the following condition holds:

$$\|Q(j\omega)\| < \frac{1}{\|\Delta(j\omega)\|} \quad \forall\, \omega \in [-\infty, \infty] \tag{11}$$

with $\|\cdot\|$ denoting the spectral norm (the largest singular value) of the corresponding matrix, or (another sufficient condition)

$$\|Q(j\omega)\|_\infty < \frac{1}{\|\Delta(j\omega)\|_\infty}, \tag{12}$$

where $\|\cdot\|_\infty$ stands for the Hardy norm [3], which is determined for any transfer matrix $\Phi(j\omega)$ as

$$\|\Phi(j\omega)\|_\infty = \sup_\omega \|\Phi(j\omega)\|. \tag{13}$$

Two main types of uncertainties (perturbation) used in the BPM are called *additive* and *multiplicative* [3]. Below we discuss only the case of additive uncertainties. The multiplicative uncertainties are analyzed analogously.

As can be seen from the matrix block diagram in Fig. 1, there are two essentially different structural blocks in the uniform system, namely, the numerical matrix of rigid cross-connections R and scalar (diagonal) transfer matrix $w(s)I$ of identical separate channels. When analyzing the uniform system robustness, it is appropriate to consider the influence of uncertainties in these two blocks separately. The case of joint perturbations can be treated by general methods [3].

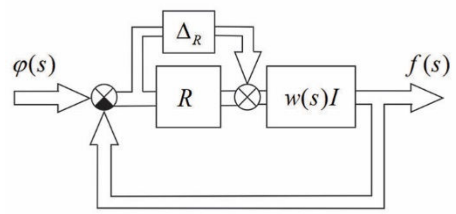

Fig. 3. Additive perturbation of the numerical matrix R

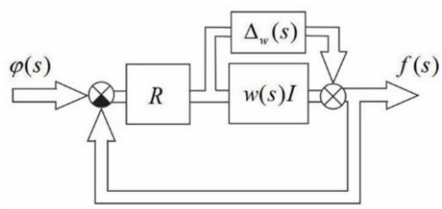

Fig. 4. Additive perturbation of the transfer matrix $w(s)I$

The matrix block diagrams in Fig. 3 and Fig. 4 represent the matrix block diagrams of the uniform system with additive uncertainties in the matrix R and the transfer matrix $w(s)I$.

Note that the perturbation Δ_R in Fig. 3 is assumed to be numerical, and the perturbation $\Delta_w(s)$ of $w(s)I$ in Fig. 4 may be generally frequency-dependent and nondiagonal.

It can be shown that the matrix $Q(j\omega)$ in Fig. 2 for the perturbed uniform system in Fig. 3 has, accounting for (3) and (6), the following form:

$$Q_R(j\omega) = -T(j\omega)R^{-1} = -Cdiag\left\{\frac{w(j\omega)}{1+\lambda_i w(j\omega)}\right\}C^{-1}. \quad (14)$$

Analogously, for the perturbed system in Fig. 4 the matrix $Q(j\omega)$ takes on the form:

$$Q_w(j\omega) = -S(j\omega)R = -Cdiag\left\{\frac{\lambda_i}{1+\lambda_i w(j\omega)}\right\}C^{-1}. \quad (15)$$

IV. Robustness analysis of uniform systems

Below, when analyzing the stability robustness of the uniform system in Fig. 1, we shall rely on the expressions (14) and (15). Our goal is to express the general robustness conditions (11) and (12) in terms of the CTFs method and to obtain simple graphical tests of stability robustness based on the characteristic gain loci of the *open-loop* uniform system.

Additive perturbation of the matrix *R*. In case of additive perturbation Δ_R of the matrix R, the expression (11) can be rewritten as

$$\left\|Q_R(j\omega)\right\| < \frac{1}{\left\|\Delta_R\right\|} \quad \forall\, \omega \in [-\infty, \infty], \quad (16)$$

where $Q_R(j\omega)$ is given by (14). Using the standard rules of matrix multiplication and norms, we have the following estimate for the upper bound of the norm $\left\|Q_R(j\omega)\right\|$:

$$\left\|Q_R(j\omega)\right\| = \left\|Cdiag\left\{\frac{w(j\omega)}{1+\lambda_i w(j\omega)}\right\}C^{-1}\right\| \le \le \nu(C)\max_i\left|\frac{w(j\omega)}{1+\lambda_i w(j\omega)}\right|, \quad (17)$$

where

$$\nu(C) = \|C\|\cdot\left\|C^{-1}\right\| \ge 1 \quad (18)$$

is the *condition number* of the modal matrix C in (3). Note that the condition number (18) is equal to one only for normal matrices R having an orthogonal canonical basis [4].

Accounting for (17), one can state that if the condition

$$\max_i\left|\frac{w(j\omega)}{1+\lambda_i w(j\omega)}\right| < \frac{1}{\nu(C)\left\|\Delta_R\right\|} \quad (19)$$

is satisfied for all frequencies ω, then the sufficient condition (16) of stability robustness of uniform system is also satisfied.

Expression (19) allows imparting a simple geometrical interpretation to the robust condition (16). If we replace the sign < in (19) by the equality sign, then after some simple algebraic manipulations that condition can be rewritten in the following form:

$$\left[\mathrm{Re}\{\lambda_i w(j\omega)\} + \frac{\alpha_i^2}{\alpha_i^2-1}\right]^2 + \left[\mathrm{Im}\{\lambda_i w(j\omega)\}\right]^2 = = \frac{\alpha_i^2}{\left(\alpha_i^2-1\right)^2}, \quad (20)$$

where

$$\alpha_i = \frac{|\lambda_i|}{\nu(C)\left\|\Delta_R\right\|}. \quad (21)$$

Geometrically, this expression determines on the complex plane of the i-th characteristic gain locus $\lambda_i w(j\omega)$ a circle with the center at the real point $c_i = -\alpha_i^2/(\alpha_i^2-1)$ with and the radius $r_i = \alpha_i/(\alpha_i^2-1)$ (Fig. 5). The sufficient condition (16) is satisfied if the circles (20) do not intersect the corresponding graphs of $\lambda_i w(j\omega)$ for all $i = 1, 2, ..., N$. For the numerical values shown in Fig. 5, the center of the circle is at the point $c_i = -1.19$ and the radius is $r_i = 0.48$. Note that for $\left\|\Delta_R\right\| = 0$, all circles (19) reduce to the critical point $-1, j0$.

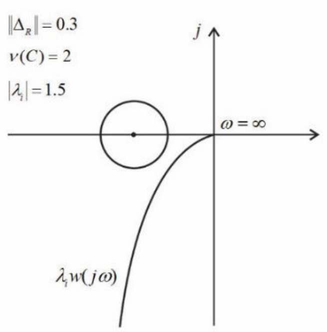

Fig. 5. Analysis of stability robustness: additive perturbations of the matrix R

Additive perturbation of the transfer matrix w(s)I. Performing analogous transformations with the sufficient condition (12) , we come to the following equation:

$$\left[\mathrm{Re}\{w(j\omega)\} + \mathrm{Re}\{1/\lambda_i\}\right]^2 + \left[\mathrm{Im}\{w(j\omega)\} + \mathrm{Im}\{1/\lambda_i\}\right]^2 = = \nu^2(C)\left\|\Delta_w(j\omega)\right\|_\infty^2. \quad (22)$$

Geometrically, it determines on the complex plane of the one hodograph of $w(j\omega)$ N circles with centers at the critical points $-1/\lambda_i$, where all circles have the same radius $r = \nu(C)\left\|\Delta_w(j\omega)\right\|_\infty$. For $N = 3$ this is illustrated in Fig. 6, where the radiuses of all circles are equal to $r = 0.51$.

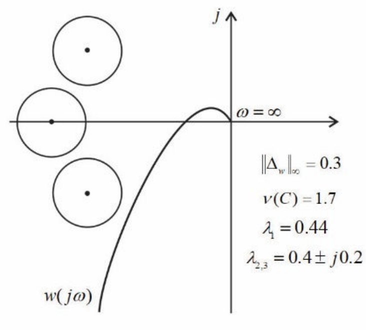

Fig. 6. Analysis of stability robustness: additive perturbations of the transfer matrix $w(s)I$)

Again, the condition (11) is satisfied if none of the circles (21) intersects the graph of $w(j\omega)$.

It is important to note that, as can be seen from (20)-(22), the radiuses of the 'forbidden' circles are proportional to the condition number $\nu(C)$ (18) of the modal matrix C. This means that the uniform systems with normal matrices R, that is systems with orthogonal canonical bases, for which $\nu(C)=1$ are more robust as compared with uniform systems with all other types of the matrix R .

It should also be noted that the presented graphical tests of robustness belong to the so-called 'very sufficient' criteria since an additional inequality in used in (17). On the other hand, the tests are very easy to use and, what is also important, they are based on the CTFs of the *open-loop* uniform systems.

V. NUMERICAL EXAMPLE

The block diagram in Fig. 7 depicts the four-dimensional linear control system of a quadcopter with the following parameters: $m=2.5\,kg$, $I_x=I_y=I_z=0.5\,kg\cdot m^2$, and identical PID-regulators in separate channels having the form [5, 6]:

$$w_R(s) = 0.0928 + \frac{0.0043}{s} + \frac{5.25}{0.1834s+1}. \quad (23)$$

Diagonal matrix J is the inertia tensor with the elements I_x, I_y, I_z.

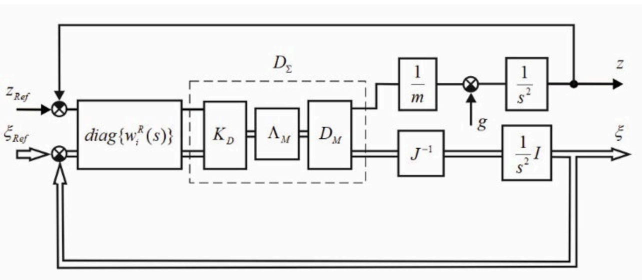

Fig. 7. Control system of quadcopter

The matrices D_M and $K_D=D_M^{-1}$ in Fig. 7 describe the rigid kinematic and artificial cross-connections between separate channels of the system and are equal to

$$D_M = \begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 \\ 0 & 0.2 & 0 & -0.2 \\ -0.2 & 0 & 0.2 & 0 \\ -0.3 & 0.3 & -0.3 & 0.3 \end{bmatrix}, \quad (24)$$

$$K_D = \begin{bmatrix} 0.25 & 0 & -2.5 & -0.83 \\ 0.25 & 2.5 & 0 & 0.83 \\ 0.25 & 0 & 2.5 & -0.83 \\ 0.25 & -2.5 & 0 & 0.83 \end{bmatrix}. \quad (25)$$

The diagonal matrix

$$\Lambda_M = I + \Delta_M, \quad (26)$$

where

$$\Delta_M = -diag\left\{\lambda_i^M\right\} \quad (0 \le \lambda_i^M < 1) \quad (27)$$

accounts for the motors' partial degradations [5]. Note that the matrix Λ_M has the form of additive perturbation of the identity matrix I . For normally functioning motors, all $\lambda_i^M=0$ and the matrix Λ_M (26) is reduced to the 4×4 identity matrix I . As a result, the matrix

$$D_\Sigma = D_M \Lambda_M K_D = D_M \Lambda_M D_M^{-1} \quad (28)$$

is also reduced to I , and the control system in Fig. 7 splits into four independent SISO channels. On the other hand, if $\Delta_M \ne I$, the cross-connections are not compensated and the system in Fig. 7 belongs to uniform control systems.

The transfer matrix of the open-loop uniform system in Fig. 7 in case of $\Lambda_M \ne I$ (and $D_M \ne I$) has the form (1), where $w(s)=w_R(s)/s^2$ and

$$R = M_\Sigma^{-1} D_\Sigma = M_\Sigma^{-1} D_M \Lambda_M D_M^{-1} = M_\Sigma^{-1} + M_\Sigma^{-1} D_M \Delta_M D_M^{-1}, \quad (29)$$

where M_Σ is a diagonal matrix with the following diagonal elements: 2.5, 0.5, 0.5, 0.5.

Application of the described procedure of analyzing robustness of uniform systems with additive perturbation of the matrix R is illustrated in Fig. 8.

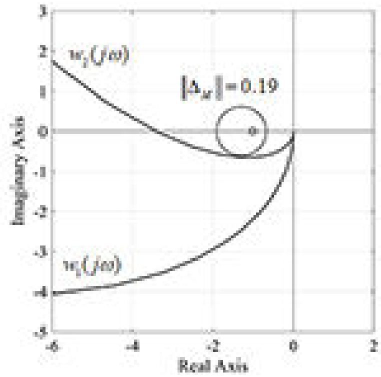

Fig. 8. Robustness analysis of the quadrotor's control system

For $\|\Lambda_M\|=0.19$, the circle (20) is tangent to hodographs of $w_2(j\omega) = w_3(j\omega) = w_4(j\omega) = 2w_R(j\omega)/(j\omega)^2$. The center of the circle is at the real point $c_2=-1.29$ and the radius is $r_2=0.613$. The corresponding circle for the transfer function $w_1(s)=0.4w_R(s)/s^2$ for the same value $\|\Lambda_M\|=0.19$ has the center at $c_1=-1.01$ and the radius $r_1=0.08$. That circle is also shown in Fig. 8. Hence, for the motor's degradations up to $\|\Lambda_M\|=0.19$, the stability of the discussed control system of the quadcopter is guaranteed. For larger values of $\|\Lambda_M\|$, the sufficient condition of robustness (16) is not satisfied, though that does not mean that the control system will become unstable.

VI. ACKNOWLEDGMENT

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