Quantified Refutation Universal System for Many-Valued Logic

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***Abstract*—A new quantified refutation proof system is introduced such that every quantified many-valued unsatisfiable formula for each version of MVL can be refuted in the described system. This proof system is based on the splitting method of variables. It is "weak" system with a "simple" proof construction strategy. However the preference for such systems lies in the possibility of proof simplification by choosing the order of splinted variables. It is also shown that the variant of this system for two-valued quantified formulas is much better by proof complexities of some formula classes than any quantified resolution systems.**

***Keywords*--many-valued propositional logic, quantified formulas, generalization of splitting method, proof complexity.**

I. Introduction

It is known that many-valued logic (MVL) was created and developed in 1920 by Łukasiewicz [1], who introduced the basic idea of additional truth degrees. In the earlier years of development, this caused some doubts about the usefulness of MVL. In the meantime, many interesting applications of MVL were found in such fields as Logic, Mathematics, Formal Verification, Artificial Intelligence, Operations Research, Computational Biology, Cryptography, Data Mining, Machine Learning, Hardware Design, Computational Biology and Medical Diagnosis. etc., therefore, the investigations in the area of MVL are very actual. The main theoretical results concern several properties of formal systems, which can present different variants of MVL and, in particular, issues on the logical completeness of defined systems. Three universal proof systems for all versions of propositional MVL were given in [2,3], where some questions, referring to the proof complexities of MVL-tautologies, are investigated as well.

While traditionally the complexity of proofs for propositional tautologies has been at the centre of research, the past two decades have witnessed a surge in proof complexity of quantified Boolean formulas (QBFs), which give not only a new class of tautologies, but also some quantifier-free tautologies can be proved simpler in any quantified system. Some interesting survey of proof complexity for QBFs is given in [4], where the complexities for some families of unsatisfiable QBFs are compared in different quantified propositional refutation proof systems.

The current research refers to the problem of constructing a modification of a universal proof system, based on the splitting method of variables, in which every unsatisfiable quantified many-valued formula for each version of MVL can be refuted in the described system.

The main notions and notations of some versions of MVL of the splitting method and of proof complexity measures are given in Preliminaries. Then the quantified many-valued formula (QMVF), the Quantified Refutation Universal System for Many-Valued Logic are described, and finally it is proved that a particular variant of this system for two-valued quantified formulas is much better by the proof complexities of some unsatisfiable formula classes than any quantified resolution system.

II. Preliminaries

2.1. Main notions of k-valued logic

Let E_k be the set $\left\{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}, 1\right\}$.We use the well-known notions of propositional formula, which is defined as usual from k-valued propositional variables with values from E_k and logical connectives $\&$, $\vee$, $\supset$, $\neg$, defined by different mode (see, for example, [4]):

(1) $\boldsymbol{p} \vee \boldsymbol{q} = max\,(p, q)$ or
(2) $\boldsymbol{p} \vee \boldsymbol{q} = \min(p + q, 1)$,
(1) $\boldsymbol{p} \& q = min\,(p, q)$ or
(2) $\boldsymbol{p} \& q = \max(\boldsymbol{p} + q - 1, 0)$

(1) $\boldsymbol{p} \supset \boldsymbol{q} = \begin{cases} 1, & for\ p \le q \\ 1 - p + q, & for\ p > q \end{cases}$ or

(2) $\boldsymbol{p} \supset \boldsymbol{q} = \begin{cases} 1, & for\ p \le q \\ q, & for\ p > q \end{cases}$

And for negation, two versions as well:
(**1**) $\neg \boldsymbol{p} = 1 - p$ or
(2) by permuting the truth values cyclically $\neg \boldsymbol{p} = (((k-1)p + 1) mod\ k)/(k-1)$.

For propositional variable $\boldsymbol{p}$ and $\boldsymbol{\delta} = \frac{i}{k-1}$ $(0 \le i \le k-1)$ we define additionally two modes of exponential function p^{σ}:
(1) $p^{\delta} = (p \supset \delta) \& (\delta \supset p)$ with (1) implication and
(2) p^{δ} as p with $(k-1)(1-\delta)$ (2) negations,

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and introduce the additional notion of formula: for every formulas A and B the expression $\boldsymbol{A^B}$ (for both modes) is also a formula.

For every propositional variable p and $\boldsymbol{\delta}=\frac{i}{k-1}$ $(0 \le i \le k-1)$ in k-valued logic p^{σ} in the sense of both exponent modes are the literals.

In the considered logics, either only 1 or all values $1/2 \le \boldsymbol{i/k-1} \le 1$ can be fixed as designated values, so a formula φ with variables $p_1, p_2, \ldots p_n$ is called 1-***k*-tautology** or ≥1/2-k-tautology if for every $\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_k^n$ assigning δ_j $(1 \le j \le n)$ to each p_j gives the value 1 or $\frac{i}{k-1}$ of φ.

2.2. Replacement rule and auxiliary relations for replacement

Replacement rules are each of the following trivial identities for a propositional formula $\boldsymbol{\psi}$:
for both variants of conjunction and both variants of disjunction
$\varphi\&0 = 0\&\varphi = 0$, $\varphi \vee 0 = 0 \vee \varphi = \varphi$,
$\varphi\&1 = 1\&\varphi = \varphi$, $\varphi \vee 1 = 1 \vee \varphi = 1$,
for (1) implication $\varphi \supset 0 = \neg\, \varphi$ with (1) negation,
$0 \supset \varphi = 1$, $\varphi \supset 1 = 1$, $1 \supset \varphi = \varphi$,
for (2) implication $1 \supset \varphi = \varphi$, $\varphi \supset 1 = 1$,
$0 \supset \varphi = 1$, $\varphi \supset 0 = \neg\, sg(\text{value of } \varphi)$,
for (1) negation $\neg(\frac{i}{k-1}) = 1 - \frac{i}{k-1}$ $(0 \le i \le k-1)$,
for (2) negation $\neg(\frac{i}{k-1}) = \frac{i+1}{k-1}$ $(0 \le i \le k-2)$, $\neg\mathbf{1} = \mathbf{0}$,
($\boldsymbol{\psi}$ with k negations) = $\boldsymbol{\psi}$.

Application of a replacement rule to a word consists of replacing its subwords, having the form of the left-hand side of one of the above identities, by the corresponding right-hand side.

In [2], the following auxiliary relations for replacements are introduced as well:
for both variants of the conjunction
$\varphi\&\frac{i}{k-1} = \frac{i}{k-1}\&\varphi \le \frac{i}{k-1}$ $(1 \le i \le k-2)$,
for both variants of disjunction
$\varphi \vee \frac{i}{k-1} = \frac{i}{k-1} \vee \varphi \ge$ value of φ $(0 \le i \le k-2)$,
for (1) implication
$\varphi \supset \frac{i}{k-1} \ge \frac{i}{k-1}$ and $\frac{i}{k-1} \supset \varphi \ge \frac{k-i+1}{k-1}$ $(1 \le i \le k-2)$,
for (2) implication
$\varphi \supset \frac{i}{k-1} \ge \frac{i}{k-1}$ $(1 \le i \le k-2)$ and
$\frac{i}{k-1} \supset \varphi \ge$ value of φ $(1 \le i \le k-1)$.

2.3. Splitting tree

Let φ be a propositional formula of k-valued logic with variables $p_1, p_2, \ldots, p_n$.

The splitting method was described at first in [5] for each formula of two-valued classical logic, and was generalized for the formulas of MVL in [3].

Let φ be some propositional formula of k-valued logic and p be one of its variables. Results of the splitting method of the formula φ by the variable p (splinted variable) are the formulas $\varphi[p^{\boldsymbol{\delta}}]$ for every $\boldsymbol{\delta}$ from the set $\left\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\right\}$, which are obtained from φ by assigning $\boldsymbol{\delta}$ to each occurrence of p and successively using replacement rules and, if, it is necessary, the auxiliary relations for replacement as well. Note that, in some cases, the formulas $\varphi[p^{\boldsymbol{\delta}}]$can remain after pointed transformation occurrences of the constant $\boldsymbol{\delta}$ as well.

The generalization of the splitting method allows us to associate with every formula φ of some tree with a root, the nodes of which are labeled by formulas and edges, labeled by literals. The root is labeled by its formula φ. If some node is labeled by the formula v and α is one of its variables, then all the k edges, which go out from this node, are labeled by one of literals $\alpha^{\boldsymbol{\delta}}$ for every $\boldsymbol{\delta}$ from the set $\left\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\right\}$, and each of k "sons" of this node is labeled by the corresponding formula $v[\alpha^{\boldsymbol{\delta}}]$. Each of the tree's leaves is labeled with some constant from the set $\left\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\right\}$. The tree, which is constructed for the formula φ by the described method, we will call splitting tree (s.t.) of φ in the future. It is obvious that by changing the order of splinted variables in the given formula φ, we can obtain different splitting trees of φ.

The universal proof system for MVL, based on the splitting method (UGSS in the future), can be defined as follows: for every formula φ, some s.t. must be constructed, and if all the tree's leaves are labeled by the value 1 (or by some value $\frac{i}{k-1} \ge 1/2$), then the formula φ is 1-k-tautology (≥1/2-k-tautology). It is obvious that the system UGSS is complete and sound.

2.4. Proof complexities

By $|\varphi|$ we denote the size of a formula φ, defined as the number of all logical signs in it. It is obvious that the full size of a formula, which is understood to be the number of all symbols is bounded by some linear function in $|\varphi|$.

The t-complexity (*l*-complexity) of s.t. is the *number (the sum of sizes) of different formulas*, with which its nodes are labeled. The t-**complexity (*l*-complexity) of the UGSS proof for MVL-tautology φ** is the value of the minimal t-complexity (***l***-complexity) of its splitting trees.

III. Main Results

The notion of the quantified many-valued formula is described, and a new quantified refutation proof system is introduced here, such that every quantified many-valued unsatisfiable formula for each version of MVL can be refuted in the described system. It is also shown that the variant of this system for two-valued quantified formulas is much better by proof complexities of some unsatisfiable formula classes than any quantified resolution systems.

3.1. Quantified many-valued formula

A QMVF is a propositional formula of MVL augmented with k-valued quantifiers ∀, ∃ that range over the values $\left\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\right\}$. In the standardized QMVFs investigated in computer science, all quantifiers appear outermost in a prefix (quantifier) and are followed by a propositional formula, called a *matrix*. The variables following after the quantifier ∀ are called *universal* variables and the variables, following after quantifier ∃ are called *existential variables*.

The quantified universal proof system for MVL, based on the splitting method (QUGSS), works as follows: for any QMVF formula φ we use the system UGSS to matrix of φ. S.t. for every QMVF tautology φ the following must hold: if for any step the splinted variable α is the universal variable of φ, then all k subtrees stuffed from the $\alpha^{\boldsymbol{\delta}}$ labeled edges for every $\boldsymbol{\delta}$ from the set $\left\{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}, 1\right\}$ must have some branch ending in one of the designated values of labeled leaves; if for any step the splinted variable α is existential variable of φ, then at least one of subtrees stuffed from the $\alpha^{\boldsymbol{\delta}}$ labeled edges must have some branch ending in one of designated values of the labeled leaves.

3.2. The quantified refutation proof system for MVL

It seems, that refutation splitting proof system can be useless, because we can use the system QUGSS to negate of the given QMVF. However, there are two form of negation in the MVL, and one of them may create some difficulties. The quantified refutation universal proof system for MVL, based on the generalized splitting method (**QRUGSS**), works as follows: for any QMVF formula φ, we use the system UGSS to the matrix of φ. S.t. for every unsatisfiable QMVF φ must be as follows: if for any step the splinted variable α is the universal variable of φ, then at least one of k subtrees stuffed from the $\alpha^{\boldsymbol{\delta}}$ labeled edges for every $\boldsymbol{\delta}$ from the set $\left\{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}, 1\right\}$ must have some branch ending in one of not designated values of the labeled leaves; if for any step the splinted variable α is the existential variable of φ, then all k subtrees stuffed from the $\alpha^{\boldsymbol{\delta}}$ labeled edges must have some branch ending in one of not designated values of the labeled leaves.

3.3. Some result of QRUGSS application

Three families of unsatisfiable QBF are given in [6]:

a) equality families of QBFs

$$\mathbf{SC_n} = \exists \mathbf{x_1} \dots \mathbf{x_n} \forall \mathbf{u_1} \dots \mathbf{u_n} \exists \mathbf{t_1} \dots \mathbf{t_n}$$
$$\Big(\mathop{\&}_{1\le i\le n} (\mathbf{x_i} \leftrightarrow \mathbf{u_i}) \supset \overline{\mathrm{ti}}\Big) \,\&\, \Big(\bigvee_{1\le i\le n} \mathbf{t_i}\Big),$$

b) $\mathbf{QParity^n} = \exists x_1 \cdots x_n \forall u \, \exists t_1 \cdots t_n$

$$(x_1 \leftrightarrow t_1) \,\&\, \mathop{\&}_{2\le i\le n} ((t_{i-1} \oplus x_i) \leftrightarrow t_i) \,\&\, (u \leftrightarrow t_n)$$

c) $\mathbf{KBKF}_{\boldsymbol{n}} = \exists x_1 y_1 \forall u_1 \cdots \exists x_n y_n \forall u_n \exists z_1 \cdots z_n,$

$(\neg x_1 \vee \neg y_1)$

$(x_i \vee u_i \vee \neg x_{i+1} \vee \neg y_{i+1}),\quad 1\le i\le n-1,$

$(y_i \vee \neg u_i \vee \neg x_{i+1} \vee \neg y_{i+1}),\quad 1\le i\le n-1,$

$(x_n \vee u_n \vee \neg z_1 \vee \cdots \vee \neg z_n),$

$(y_n \vee \neg u_n \vee \neg z_1 \vee \cdots \vee \neg z_n),$

$(u_i \vee z_i),\quad 1\le i\le n,$

$(\neg u_i \vee z_i),\quad 1\le i\le n$

As it is mentioned in [6], all the above formulas are exponentially hard for quantified system QU-Resolution (i.e., they require proofs of exponential size).

The following statement is proved.

Theorem: If φ_n are formulas $\mathbf{SC_n}$, $\mathbf{QParity_n}$ or $\mathbf{KBKF_n}$, then in the system QUGSS for two-valued formulas $t(\varphi_n) = \theta(n)$ and $l(\varphi_n) = \theta(n^2)$.

Proof sketch:

a) If for all n φ_n is $\mathbf{SC_n}$, then we can choose for the splinted variables such sequence, which allows us to construct for φ_n some s.t. $T(\varphi_n)$ with the following complexities
$t(T(\varphi_1))=6$ and $t(T(\varphi_n)) = t(T(\varphi_{n-1}))+4$.

b) If for all n φ_n is $\mathbf{QParity_n}$, then we can choose for the splinted variables such sequence, which allows us to construct for φ_n some s.t. $T(\varphi_n)$ with the following complexities
$t(T(\varphi_1))=7$,
$t(T(\varphi_n)) = t(T(\varphi_{n-1}[u^0]))+t(T(\varphi_{n-1}[u^i]))+8$
and $\forall\ 2\le i\le n-1$
$t(T(\varphi_i[u^0]))+t(T(\varphi_i[u^i]))=$
$t(T(\varphi_{i-1}[u^0]))+t(T(\varphi_{i-1}[u^i]))+8$.

c) If for all n φ_n is $\mathbf{KBKF_n}$, then we can choose for the splinted variables such sequence, which allows us to construct for φ_n some s.t. $T(\varphi_n)$ with the complexity $t(T(\varphi_n)) \le 11(n-2)+9$.

Take into consideration that the longest formula of proof is the matrix of the formula φ_n with size $c_1 n$ for some constant c_1 and the longest branch in any s.t. of the matrix of the formula φ_n must have $c_2 n$ nodes for some constant c_2, we obtain the upper and lower bounds for $t(\varphi_n)$ and $l(\varphi_n)$.

As the introduced system has a simple strategy for constructing proofs, besides its mathematical significance, it can have practical applications in many areas.

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