On Hamiltonian Bypasses in Digraphs

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***Abstract*—A Hamiltonian path in a digraph D, in which the initial vertex dominates the terminal vertex, is called a Hamiltonian bypass. Let D be a 2-strong digraph of order $p \geq 3$ and let z be some vertex of D. Suppose that every vertex of D other than z has a degree of at least p. We introduce and study a conjecture, which claims that there exists a smallest integer k such that if $d(z) \geq k$, then D contains a Hamiltonian bypass. In this paper, we prove: If D is Hamiltonian or z has a degree greater than $(p-1)/3$, then D contains a Hamiltonian bypass. This result improves the result of Benhocine (J. of Graph Theory, 8, 1984) and the result of the author (Math. Problems of Computer Science, 54, 2020). We also suggest some conjectures and problems.**

***Keywords*— Digraph, cycle, Hamiltonian cycle, Hamiltonian bypass.**

I. Introduction and Terminology

We shall assume that the reader is familiar with the standard terminology on digraphs, and for the terminology and notation not defined in this paper, the reader is referred to [1]. In this paper, we consider finite digraphs without loops and multiple arcs, which may contain opposite arcs with the same end-vertices, i.e., a cycle of length two. Each cycle and path is assumed to be simple and directed. A cycle (path) in a digraph D that passes through all the vertices of D is called *Hamiltonian*. A digraph containing a Hamiltonian cycle is called a *Hamiltonian digraph*. One of the fundamental and most studied problems in digraph theory is to find sufficient conditions for a digraph to contain a Hamiltonian path of a certain type. A Hamiltonian path in a digraph D is called a Hamiltonian bypass if its initial vertex dominates its terminal vertex. There are many sufficient conditions for the existence of a Hamiltonian cycle in digraphs (see, e.g., [2] - [9]). It is natural to consider an analogous problem for the existence of a Hamiltonian bypass.

It was proved in [10] - [15] that a number of sufficient conditions for a digraph to be Hamiltonian are also sufficient for a digraph to contain a Hamiltonian bypass (except for some exceptional digraphs, which are characterised). They include Theorems 1.1-1.4.

Theorem 1.1 (Benhocine [10]). Let D be a 2-strong digraph of order p with a minimum degree of at least $p-1$. Then D contains a Hamiltonian bypass, except for some exceptional digraphs, which are characterised.

Theorem 1.2 (Benhocine [10]). Every digraph D of order p and a minimum degree of at least p contains a Hamiltonian bypass.

Theorem 1.3 (Darbinyan [12]). Let D be a strong digraph of order $p \geq 3$. Suppose that $d(x) + d(y) \geq 2p - 2$ for every pair of non-adjacent vertices x, y of $V(D)$. Then D contains a Hamiltonian bypass, except for some exceptional digraphs, which are characterised.

Theorem 1.4 (Darbinyan [14]). Let D be a strong digraph of order $p \geq 4$. Suppose that for each triple of vertices x, y, z such that x and y are non-adjacent vertices: If there is no arc from x to z, then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3p - 2$. If there is no arc from z to x, then $d(x) + d(y) + d^-(x) + d^+(z) \geq 3p - 2$. Then D contains a Hamiltonian bypass, unless D is a single tournament with five vertices.

Let D be a 2-strong digraph of order $p \geq 3$, in which $p-1$ vertices have a degree of at least p. In [13], the author proved that if D is Hamiltonian or its minimum degree is more than $0.4(p-1)$, then D has a Hamiltonian bypass. In this paper, we prove the following theorem.

Theorem 1.5. Let D be a 2-strong digraph of order $p \geq 3$ and let z be some vertex of D. Suppose that $d(x) \geq p$ for each vertex $x \in V(D) \setminus \{z\}$. If D is Hamiltonian or $d(z) > (p-1)/3$, then D has a Hamiltonian bypass.

II. Sketch of the Proof of Theorem 1.5

In the proof of Theorem 1.5, we used the following results (Lemmas 2.1-2.3) from [13] and Theorem 2.1 from [16].

Lemma 2.1 (Claim 1 and Case 1 in [13]). Let D be a digraph of order $p \geq 5$ and z be some vertex of D. Suppose that every vertex of $V(D) \setminus \{z\}$ has a degree of at least p. If D contains a cycle of length at least $p-2$ through z, then D contains a Hamiltonian bypass.

Lemma 2.2 (Claim 4 and Proposition 1 in [13]). Let D be a 2-strong digraph of order $p \geq 3$ and z be some vertex of D. Suppose that every vertex of $V(D) \setminus \{z\}$ has a degree of at least p. If a longest path through z, say $P = x_1x_2 \ldots x_l$, has length at least $p-3$ and $x_1x_l \in A(D)$, then D contains a Hamiltonian bypass.

Lemma 2.3 (Lemma 4 in [13]). Let D be a 2-strong digraph of order $p \geq 3$ and let z be some vertex of $V(D)$. Suppose that every vertex of $V(D) \setminus \{z\}$ has a degree of at least p. If

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a longest cycle through z has length $p-3$, then D contains a Hamiltonian bypass.

We also present a new, significantly shorter proof of Lemma 2.3 than that given in [9].

Theorem 2.1 (Darbinyan [16]). Let D be a strong digraph of order $p \geq 3$. If $p-1$ vertices of $V(D)$ have a degree of at least p, then D is Hamiltonian or contains a cycle of length $p-1$ that contains all the vertices with a degree of at least p.

To prove the main result, we first proved the following basic lemma.

Lemma 2.4. Let D be a 2-strong digraph of order $p \geq 3$ and let z be some vertex of D. Suppose that every vertex of $V(D) \setminus \{z\}$ has a degree of at least p. If a longest cycle through z has length $p-4$, then D contains a Hamiltonian bypass.

The following results follow from Theorem 1.5.

Corollary 1 (Benhocine [10]). Every strong digraph D of order $p \geq 3$ and with minimum degree at least p contains $D(p, 2)$.

Corollary 2 (Darbinyan [13]). Let D be a 2-strong digraph of order $p \geq 3$ and let z be some vertex of D. Suppose that $d(x) \geq p$ for each vertex $x \in V(D) \setminus \{z\}$. If D is Hamiltonian or $d(z) > 0.4(p-1)$, then D has a Hamiltonian bypass.

III. Conclusion

In the current article, we investigated the existence of a Hamiltonian bypass in 2-strong digraphs of order p, in which $p-1$ vertices have degrees at least p. We proved that if such digraphs are Hamiltonian or have the minimum degree greater than $(p-1)/3$, then they contain a Hamiltonian bypass. In [16], a 2-strong non-Hamiltonian digraph of order n was constructed, in which $n-1$ vertices have a degree of at least n and the remaining vertex has a degree equal to four. It is straightforward to verify that this digraph possesses a Hamiltonian bypass. According to this argument, we believe the following conjecture to be true:

Conjecture 1. Let D be a 2-strong digraph of order p. If $p-1$ vertices in $V(D)$ have degrees at least p, then D contains a Hamiltonian bypass.

We also propose the following two conjectures.

Conjecture 2. Let D be a strong digraph of order p such that for every distinct pair of nonadjacent vertices x, y, and, w, z we have $d(x) + d(y) + d(w) + d(z) \geq 4p - 4$. Then D has a Hamiltonian bypass, except for certain digraphs.

For arbitrary integers $n \geq 3$ and $q \in [2, n]$, $D(n, q)$ denotes the digraph of order n obtained from a directed cycle of length n, by changing the orientation of $q-1$ consecutive arcs. Benhocine and Wojda [11] proved that any digraph D of order p that satisfies the condition that the sum of the degrees for any two non-adjacent vertices is at least $2p$, contains $D(n, 2)$ for every $n \in [3, p]$, except for certain digraphs. We are sure that if we replace $2p$ with $2p-1$ in this result, then the result will be correct again (Conjecture 3).

Conjecture 3. Let D be a strong digraph of order p, which satisfies the condition that the sum of the degrees for any two nonadjacent vertices is at least $2p-1$. Then D contains $D(n, 2)$ for each $n \in [3, p]$, except for certain digraphs.

Note that the author [12] proved that, under the conditions of Conjecture 3, a digraph contains a $D(p, 3)$.

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