About Some Large Cap Sets

Iskandar Karapetyan
Institute for Informatics and Automation Problems of NAS RA
Yerevan, Armenia
e-mail: isko@iiap.sci.am

Karen Karapetyan
Institute for Informatics and Automation Problems of NAS RA
Yerevan, Armenia
e-mail: karen-karapetyan@iiap.sci.am

***Abstract*—A cap set in a projective or affine geometry over a finite field is a set of points no three of which are collinear. We construct complete cap sets with sizes 274432, 13991936, and 30294016 in affine geometries $AG(16,3)$, $AG(21,3)$, and $AG(22,3)$, respectively.**

Keywords—Affine geometry, points, cap set, complete cap set.

I. INTRODUCTION

The cap set problem asks how large a subset of affine geometry $AG(n,3)$ can be and contain no lines - that is, no three points $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ such that $\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}=\mathbf{0}\ (mod\ 3)$. The problem was motivated by the design of statistical experiments (combinatorial design) [1], arithmetic progressions in the set of prime numbers [2], card game set [3], coding theory [4], etc. Note that the problem of determining the minimum size of a complete cap set in a given projective space is of particular interest in coding theory. If we write the points of the cap set as columns of a matrix, we obtain a matrix in which every three columns are linearly independent, hence the generator matrix of a linear orthogonal array of strength three. This matrix is a check matrix of a linear code with a minimum distance greater than three. More generally, the main problem in the theory of cap sets is to find the minimal and maximal sizes of complete cap sets in projective geometry $PG(n,q)$ or in affine geometry $AG(n,q)$. Many authors have noted that determining the exact value of the minimum and maximum cardinality of cap sets in the projective geometry $PG(n,q)$ or in the affine geometry $AG(n,q)$ seems to be a very hard problem. There are some well-known constructions (product [5] and doubling [6]), which allow us to create large high-dimensional cap sets based on large low-dimensional cap sets. In this paper, we consider the problem of constructing complete cap sets in affine geometry $AG(n,3)$ over the finite field $F_3=\{0,1,2\}$. A cap set is called complete when it cannot be extended to a larger one. Let us denote the size of the largest cap set in $AG(n,q)$ and $PG(n,q)$ by $c_{n,q}$ and by $c'_{n,q}$, respectively. Presently, only the following exact values are known: $c_{n,2}=c'_{n,2}=2^n$, $c_{2,q}=c'_{2,q}=q+1$ if q is odd, $c_{2,q}=c'_{2,q}=q+2$ if q is even, and $c_{3,q}=q^2$, $c'_{3,q}=q^2+1$ [1,7]. Apart from these general results, the exact values are known in the following cases: $c_{4,3}=c'_{4,3}=20$ [8], $c'_{5,3}=56$ [9], $c_{5,3}=45$ [10], $c_{4,4}=40$, $c'_{4,4}=41$ [11], $c_{6,3}=112$ [12]. In the other cases, only lower and upper bounds on the sizes of cap sets in $AG(n,q)$ and $PG(n,q)$ are known [13]. In particular, it has been proven that $c_{7,3}\geq 236$ [14], and using a computer search $c_{8,3}\geq 512$ [15]. Also, using a computer search, the following upper bounds on the maximum size of cap sets in dimensions seven to ten were proven in [16]: $c_{7,3}\leq 291$, $c_{8,3}\leq 771$, $c_{9,3}\leq 2070$, and $c_{10,3}\leq 5619$, respectively.

In this paper, we construct complete cap sets with sizes 274432, 13991936, and 30294016 in affine geometries $AG(16,3)$, $AG(21,3)$, and $AG(22,3)$, respectively. The constructed cap sets are more powerful than those that can be obtained from the previously known ones using the product and doubling operations.

II. NOTATIONS, DEFINITIONS, AND KNOWN REZULTS

Note that three points $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ in affine geometry $AG(n,3)$ are collinear if they are affinely dependent or, equivalently, $\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}=\mathbf{0}\ (mod\ 3)$. Therefore, if C_n is a cap set in $AG(n,3)$, then $\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}\neq\mathbf{0}\ (mod\ 3)$ for every triple of distinct points $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\in C_n$. For the point $\boldsymbol{x}\in AG(n,3)$, let's denote

$$\boldsymbol{x}(0)=\{i\ |x_i=0, i\in[1,n]\},$$
$$\boldsymbol{x}(1)=\{i\ |x_i=1, i\in[1,n]\},$$
$$\boldsymbol{x}(2)=\{i\ |x_i=2, i\in[1,n]\},$$
$$X(\boldsymbol{x})=\{\boldsymbol{\alpha}|\boldsymbol{\alpha}\in AG(n,3), \boldsymbol{\alpha}(0)=\boldsymbol{x}(0)\}.$$

Further, let

$$B_n=\{\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_n)\ |\alpha_i=1,2\},$$
$$B'_n=\{\boldsymbol{\alpha}\in B_n|\ |\boldsymbol{\alpha}(1)|\ is\ odd\},$$
$$B''_n=\{\boldsymbol{\alpha}\in B_n|\ |\boldsymbol{\alpha}(1)|\ is\ even\}.$$

In 2015, K. Karapetyan [17] introduced the concept of P_n-set, which we use in our research. Notice that the set of points

https://doi.org/10.51408/csit2025_23

$A \subseteq AG(n,3)$ is called a P_n-set if it satisfies the following two conditions:

(i) for any two distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A$, there exists i such that $\alpha_i = \beta_i = 0$, where $1 \le i \le n$,
(ii) for any triple of distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in A$, $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \neq \mathbf{0}(mod\ 3)$.

Throughout this article, the notation P_n (P_{n_i}) will mean P_n-set in $AG(n,3)$ (P_{n_i}-set in $AG(n_i,3)$). We call P_n to be complete when it cannot be extended to a larger one. The set P_n is called odd if $|\boldsymbol{\alpha}(0)|$ is odd for every point $\boldsymbol{\alpha} \in P_n$ and even if $|\boldsymbol{\alpha}(0)|$ is even for every point $\boldsymbol{\alpha} \in P_n$. The set P_n is called b-saturated if $X(\boldsymbol{\alpha}) \subseteq P_n$ for every point $\boldsymbol{\alpha} \in P_n$, where $b = 1, 2$. We will define the concatenation or direct product of the sets in the following way. Let $A \subset AG(n,3)$ and $B \subset AG(m,3)$. Form a new set $AB \subset AG(n+m,3)$ consisting of all points $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n,\ \alpha_{n+1}, \cdots, \alpha_{n+m})$, where $\boldsymbol{\alpha}^1 = (\alpha_1, \cdots, \alpha_n) \in A$ and $\boldsymbol{\alpha}^2 = (\alpha_{n+1}, \cdots,\ \alpha_{n+m}) \in B$. Similarly, one can define the concatenation of the points for any number of sets. The next two theorems will introduce constructions to obtain a cap sets in higher dimensions by using known cap sets in lower dimensions. Theorem A states a simplified version of the general product construction theorem first stated by Mukhopadhyay in [6] and reformulated by Edel and Bierbrauer [5]. Theorem B, the doubling construction, is a special case of the general product construction. In this work, we will need the following theorems to construct some large complete cap sets.

Theorem A [5] (Product construction): Let $A \subseteq AG(n,3)$ and $B \subseteq AG(m,3)$ be cap sets. Then $AB \subset AG(n+m,3)$ is a cap set.

Theorem B [6] (Doubling construction): Let $A \subseteq PG(n,3)$ be a cap set. Then there is a cap set in $AG(n+1,3)$ of size $2\,|A|$.

Theorem C [18]: The set $A \subseteq AG(n,3)$ is a b-saturated and complete P_n-set if and only if it satisfies the following two conditions:

i) for any two distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A$, there exists i such that $\alpha_i = \beta_i = 0$, where $1 \le i \le n$,

iii) for any triple of distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in A$, $\boldsymbol{\alpha}(0) = \boldsymbol{\beta}(0) = \boldsymbol{\gamma}(0)$ or for two of them, say for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, there exists i such that $\alpha_i = \beta_i = 0$ and $\gamma_i \neq 0$, where $1 \le i \le n$.

Theorem D [17]: If the sets P_n and P_m are complete and b-saturated, then the set $C_{n+m} = P_n B_m \cup B_n P_m$ is a complete cap set in $AG(n+m,3)$, where n and m are any integers.

Corollary 1: For any natural number n, $c_{n+1,3} \ge 2|P_n| + |B_n|$.

Theorem E [18]: If P_n is a b-saturated, complete, and odd set, then $C_n = P_n \cup B_n'$ ($C_n = P_n \cup B_n''$) is a complete cap set.

Corollary 2: $c_{6,3} \ge 112$ [12, 14], $c_{11,3} \ge 5504$.

For the given three sets P_{n_1}, P_{n_2}, and P_{n_3}, form the following set $P_{n_1}P_{n_2}B_{n_3} \cup P_{n_1}B_{n_2}P_{n_3} \cup B_{n_1}P_{n_2}P_{n_3}$. It is known that the formed set is a P_n-set [17], where $n = \sum_1^3 n_i$ and n_1, n_2, n_3 are any integers.
Theorem F («three» construction) [17, 19]: The following recurrence relation $P_n = P_{n_1}P_{n_2}B_{n_3} \cup P_{n_1}B_{n_2}P_{n_3} \cup B_{n_1}P_{n_2}P_{n_3}$, with the initial sets $P_1 = \{(0)\}$, $P_2 = \{(0,1),(0,2)\}$ gives complete and b-saturated P_n –set, where $n = \sum_{j=1}^3 n_j$.

Theorem G [18]: Let $P_{n_1}, P_{n_2}, P_{n_3}$ be b-saturated and complete sets. If two of the sets $P_{n_1}, P_{n_2}, P_{n_3}$, say P_{n_1}, P_{n_2}, are odd, then $C_n = P_n \cup B_{n_1}'B_{n_2}'B_{n_3}$ is a complete cap set, where $P_n = P_{n_1}P_{n_2}B_{n_3} \cup P_{n_1}B_{n_2}P_{n_3} \cup B_{n_1}P_{n_2}P_{n_3}$, $n = \sum_1^3 n_i$ and n_1, n_2, n_3 are any integers.

Corollary 3: $c_{15,3} \ge 120832$, $c_{10,3} \ge 2240$.

Proof: Let's consider the set $P_{15} = P_{6+6+3}$, which is obtained by replacing n_1, n_2, n_3 in Theorem F («three» construction) with 6, 6, 3, respectively. Therefore, $P_{15} = P_{6+6+3} = P_6P_6B_3 \cup P_6B_6P_3 \cup B_6P_6P_3$. It is easy to see that the three sets $P_6P_6B_3, P_6B_6P_3, B_6P_6P_3$, are pairwise disjoint. Since $|P_3| = 6$, $|P_6| = 80$, $|B_3| = 8$, $|B_6'| = 32$ and $|B_6| = 64$. Then $|\,P_{15}\,| = 80^2 * 8 + 2 * 80 * 64 * 6 = 112640$ and $|\,B_6'B_6'B_3\,| = 8 * 1024$. Since $(P_6P_6B_3 \cup P_6B_6P_3 \cup B_6P_6P_3) \cap B_6'B_6'B_3 = \emptyset$ and P_6 is an odd set, it follows from Theorem G that $c_{15,3} \ge 112640 + 8192 = 120832$. Similarly, to prove the second inequality, let us consider the set $P_{10} = P_{1+6+3}$. It is easy to see that $|P_{10}\,| = 1984$. Because the sets P_1 and P_6 are odd, then from the same Theorem G it follows that $|C_{10}| = |P_{10} \cup B_1'B_6'B_3| = 2240$. Therefore, $c_{10,3} \ge 2240$ [18]. The last inequality also follows from Corollary 1. Corollary is proved.

Note that the obtained lower bound 120832 for $c_{15,3}$ is equal to the size of the product cap sets with sizes 236 and 512 constructed in [14] and [15], respectively.

For the given six sets $P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}$ and P_{n_6}, form the following ten sets:

$$A_1 = P_{n_1}P_{n_2}P_{n_3}B_{n_4}B_{n_5}B_{n_6}, A_2 = P_{n_1}P_{n_2}B_{n_3}B_{n_4}B_{n_5}P_{n_6},$$
$$A_3 = P_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}B_{n_6}, A_4 = B_{n_1}P_{n_2}P_{n_3}P_{n_4}B_{n_5}B_{n_6},$$
$$A_5 = B_{n_1}B_{n_2}P_{n_3}P_{n_4}B_{n_5}P_{n_6}, A_6 = B_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}P_{n_6},$$
$$A_7 = B_{n_1}P_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}, A_8 = B_{n_1}P_{n_2}B_{n_3}B_{n_4}P_{n_5}P_{n_6},$$
$$A_9 = P_{n_1}B_{n_2}B_{n_3}P_{n_4}B_{n_5}P_{n_6}, A_{10} = P_{n_1}B_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}.$$

Theorem H («six» construction) [19]: The following recurrence relation $P_n = \bigcup_{i=1}^{10} A_i$, with the initial sets $P_1 = \{(0)\}$, $P_2 = \{(0,1),(0,2)\}$ gives complete and b-saturated P_n-sets, where $n = \sum_{i=1}^{6} n_i$ and $n_1, n_2, n_3, n_4, n_5, n_6$ are any integers.

Theorem I [18]: Let $P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}$ and P_{n_6} be b-saturated and complete sets. If three of the sets $P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}, P_{n_6}$, say P_{n_1}, P_{n_2} and P_{n_3}, are odd, then $C_n = P_n \cup B'_{n_1}B'_{n_2}B'_{n_3}B_{n_4}B_{n_5}B_{n_6}$ is a complete cap set, where $P_n = \cup_1^{10} A_i, n = \sum_1^6 n_i$ and n_1, n_2, n_3, n_4, n_5, n_6 are any integers.

III. Main Results

Claim: If $x, y, z \in F_3$, then $x+y+z \equiv 0\ (mod\ 3)$ if and only if $x = y = z$ or they are pairwise distinct numbers.

Theorem 1: $c_{16,3} \geq 274432$ and $c_{21,3} \geq 13991936$.

Proof: First, let's consider the set $P_{16} = P_{6+6+1+1+1+1}$, which is obtained by replacing n_1, n_2, n_3, n_4, n_5, n_6 in Theorem H («six» construction) with 6, 6, 1, 1, 1, 1, respectively. Then from the same Theorem H it follows that this is a b-saturated and complete P_{16}-set, where

$$A_1 = P_6P_6P_1B_1B_1B_1, A_2 = P_6P_6B_1B_1B_1P_1,$$
$$A_3 = P_6B_6P_1B_1P_1B_1, A_4 = B_6P_6P_1P_1B_1B_1,$$
$$A_5 = B_6B_6P_1P_1B_1P_1, A_6 = B_6B_6P_1B_1P_1P_1,$$
$$A_7 = B_6P_6B_1P_1P_1B_1, A_8 = B_6P_6B_1B_1P_1P_1,$$
$$A_9 = P_6B_6B_1P_1B_1P_1, A_{10} = P_6B_6B_1P_1P_1B_1.$$

It is easy to see that all ten $A_1, A_2, \ldots, A_{10}$ sets are pairwise disjoint, i.e. $A_i \cap A_j = \emptyset$, where $i, j \in [1, 10]$. Since $|P_1| = 1$, $|B_1| = 2$, then $|P_6| = 80$ and $|B_6| = 64$. Therefore
$|P_{16}| = 2* 80^2 * 8+2 * 80 * 64 * 4 + 4 * 80 * 64 * 4+2 * 64^2 * 2= 241664$.

It is not difficult to check that every point of the set $P_{16} = P_{6+6+1+1+1+1}$ has exactly three, or five, or seven zero coordinates. Therefore, P_{16} is an odd set. Obviously, $P_{16} \cap B'_{16} = \emptyset$ and $|B'_{16}| = 32768$. Therefore, from Theorem E it follows that $P_{16} \cup B'_{16}$ is a complete cap set and, hence, $c_{16,3} \geq |P_{16} \cup B'_{16}| =274432$.

The proof of the second part of the theorem is similar to the first one. Consider the set $P_{21} = P_{6+6+6+1+1+1}$, which is obtained, as above, by replacing $n_1, n_2, n_3, n_4, n_5, n_6$ with 6, 6, 6, 1, 1, 1, respectively. Again, Theorem H implies that the obtained set is a b-saturated and complete P_{21}-set, where

$$A_1 = P_6P_6P_6B_1B_1B_1, A_2 = P_6P_6B_6B_1B_1P_1,$$
$$A_3 = P_6B_6P_6B_1P_1B_1, A_4 = B_6P_6P_6P_1B_1B_1,$$
$$A_5 = B_6B_6P_6P_1B_1P_1, A_6 = B_6B_6P_6B_1P_1P_1,$$
$$A_7 = B_6P_6B_6P_1P_1B_1, A_8 = B_6P_6B_6B_1P_1P_1,$$
$$A_9 = P_6B_6B_6P_1B_1P_1, A_{10} = P_6B_6B_6P_1P_1B_1,$$

and all ten $A_1, A_2, \ldots, A_{10}$ sets are pairwise disjoint, i.e., $A_i \cap A_j = \emptyset$, $i, j \in [1, 10]$. As already mentioned above $|P_6| = 80$, $|P_1| = 1$, $|B_1| = 2$ and $|B_6| = 64$, therefore,
$|P_{21}| = 80^3 * 8+12 * 80^2 * 64 + 12 * 64^2 * 80= 12943360$. Each point of the set $P_{21} = P_{6+6+6+1+1+1}$ has exactly five, seven, or nine zero coordinates. Therefore, P_{21} is an odd set. Since $P_{21} \cap B'_{21} = \emptyset$ and $|B'_{21}| = 1048576$, it follows from Theorem E that $c_{21,3} \geq |P_{21} \cup B'_{21}| =13991936$. Theorem is proved.

Note that the resulting cap sets are more powerful than those that can be obtained from the previously known ones using the product operation. Using the idea of the proof of Theorem 3 [18], we can construct cap sets of sizes 569600 and 28698112 in $AG(17,3)$ and $AG(22,3)$, respectively. But product operation and Corollary 2 imply that $c_{17,3} \geq 5504*112 = 616448$ and $c_{22,3} \geq 5504*5504 = 30294016$. It is also clear that to construct more powerful cap sets, new methods of constructing P_n-sets, especially odd P_n-sets, are needed.

The following three theorems are analogues of Theorems E, G, and I, respectively. Previous theorems, using odd P_n-sets, give cap sets in $AG(n,3)$, but these theorems, using even P_n-sets, give cap sets in $AG(n+1,3)$.

Lemma: If $|\boldsymbol{\alpha}(0)|$ is even, then $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \neq 0(mod\ 3)$ for any two points $\boldsymbol{\beta} \in B'_n$ and $\boldsymbol{\gamma} \in B''_n$.

Proof: Let's prove this by contradiction. Suppose that there are two points $\boldsymbol{\beta} \in B'_n$ and $\boldsymbol{\gamma} \in B''_n$ such that $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} = 0(mod\ 3)$. Denote $|\boldsymbol{\alpha}(0)| = 2m$, $l = |\boldsymbol{\alpha}(1)|$ and $k = |\boldsymbol{\beta}(1) \cap \boldsymbol{\alpha}(0)|$. Without loss of generality, let's assume that the first $2m$ coordinates of the point $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ are equal to zero, i.e., $\alpha_1 = \cdots = \alpha_{2m} = 0$. It is clear that the equality $\alpha_i + \beta_i + \gamma_i = 0(mod\ 3)(i \in [1,n])$ implies that $\gamma_i = 2$ for each $i \in \boldsymbol{\beta}(1) \cap \boldsymbol{\alpha}(0)$ and $\gamma_i = 1$ for each $i \in [1,2m] \backslash (\boldsymbol{\alpha}(0) \cap \boldsymbol{\beta}(1))$. Next, from Claim it follows that $\alpha_j = \beta_j = \gamma_j$ for each $j \in [1,\ n] \setminus \boldsymbol{\alpha}(0)$. If k is odd, then from the fact that $\beta \in B'_n$ it follows that l must be even. But then $|\boldsymbol{\gamma}(1)| = 2m - k + l$, which is odd and contradicts the supposition that $\boldsymbol{\gamma} \in B''_n$. Therefore, k must be even. Hence, from the fact that $\boldsymbol{\beta} \in B'_n$ it follows that l must be odd. But then $|\boldsymbol{\gamma}(1)| = 2m - k + l$, which is again odd. The resulting contradiction completes the proof. The lemma is proved.

Theorem 2: If P_n is a b-saturated and even set, then $C_n = P_n\{(0)\} \cup B'_n\{(1)\} \cup B''_n\{(2)\}$ is a cap set in $AG(n+1,3)$.

Proof: Let us carry out the proof by contradiction. Suppose that C_n is not a cap set. Therefore, there are a triple of distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in C_n$ such that $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \equiv 0(mod\ 3)$, where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n,\ \alpha_{n+1})$, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n,\ \beta_{n+1})$, $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_n,\ \gamma_{n+1})$. The following three cases are possible.

Case 1. $\alpha_{n+1} = \beta_{n+1} = \gamma_{n+1} = 0$. Then, obviously, $(\alpha_1, \cdots, \alpha_n)$, $(\beta_1, \cdots, \beta_n)$, $(\gamma_1, \cdots, \gamma_n) \in P_n$ and from condition (ii) it follows that $(\alpha_1, \cdots, \alpha_n) + (\beta_1, \cdots, \beta_n) + (\gamma_1, \cdots, \gamma_n) \neq \mathbf{0}(mod\ 3)$. Therefore, $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \neq \mathbf{0}(mod\ 3)$.

Case 2. $\alpha_{n+1} = \beta_{n+1} = \gamma_{n+1} = x$, where $x = 1$ or $x = 2$. Suppose that $x = 1$. Now then $(\alpha_1, \cdots, \alpha_n)$, $(\beta_1, \cdots, \beta_n)$, $(\gamma_1, \cdots, \gamma_n) \in B'_n$. Then Claim implies that $\alpha_i = \beta_i = \gamma_i$, where $i \in [1, n]$. Therefore, $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma}$. For $x = 2$, the proof is similar.

Case 3. α_{n+1}, β_{n+1}, γ_{n+1} are pairwise distinct numbers. Without loss of generality, we can assume that $\alpha_{n+1} = 0$, $\beta_{n+1} = 1$, $\gamma_{n+1} = 2$. Therefore, $\boldsymbol{\alpha} \in P_n\{(0)\}, \boldsymbol{\beta} \in B'_n\{(1)\}, \boldsymbol{\gamma} \in B''_n\{(2)\}$. Then it follows from the lemma that $(\alpha_1, \cdots, \alpha_n) + (\beta_1, \cdots, \beta_n) + (\gamma_1, \cdots, \gamma_n) \neq 0(mod\ 3)$. Therefore, $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \neq 0(mod\ 3)$. Theorem 1 is proved.

The proofs of the next two theorems are similar to those given above. The proof for the first two cases is the same. To prove the case 3, we represent every point $\boldsymbol{\alpha} \in P_n$ as $\boldsymbol{\alpha} = \boldsymbol{\alpha}^1\boldsymbol{\alpha}^2\boldsymbol{\alpha}^3$ ($\boldsymbol{\alpha} = \boldsymbol{\alpha}^1\boldsymbol{\alpha}^2\boldsymbol{\alpha}^3\boldsymbol{\alpha}^4\boldsymbol{\alpha}^5\boldsymbol{\alpha}^6$), where $\boldsymbol{\alpha}^1 = \left(\alpha_1, \cdots, \alpha_{n_1}\right)$, $\boldsymbol{\alpha}^2 = \left(\alpha_{n_1+1}, \cdots, \alpha_{n_1+n_2}\right)$, $\boldsymbol{\alpha}^3 = \left(\alpha_{n_1+n_2+1}, \cdots, \alpha_n\right)$ ($\boldsymbol{\alpha}^i = \left(\alpha_{\sum_{j=1}^{i-1} n_j+1}, \dots, \alpha_{\sum_{j=1}^{i} n_j}\right)$, and $i \in [1, 6]$). Then Theorem F (Theorem H) implies that $\boldsymbol{\alpha}^1 \in P_{n_1}$ or $\boldsymbol{\alpha}^2 \in P_{n_2}$ ($\boldsymbol{\alpha}^1 \in P_{n_1}$ or $\boldsymbol{\alpha}^2 \in P_{n_2}$ or $\boldsymbol{\alpha}^3 \in P_{n_3}$). Suppose that $\boldsymbol{\alpha}^i \in P_{n_i}$, where $i \in [1, 2]$ ($i \in [1, 3]$). Since $P_{n_1}, P_{n_2} (P_{n_1}, P_{n_2}, P_{n_3})$ are odd, then Lemma implies that $\boldsymbol{\alpha}^i + \boldsymbol{\beta}^i + \boldsymbol{\gamma}^i \neq 0(mod\ 3)$, where $\boldsymbol{\beta} = \boldsymbol{\beta}^1\boldsymbol{\beta}^2\boldsymbol{\beta}^3 \in B'_{n_1}B'_{n_2}B_{n_3}, \boldsymbol{\gamma} = \boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 \in B''_{n_1}B''_{n_2}B_{n_3}$ ($\boldsymbol{\beta} = \boldsymbol{\beta}^1\boldsymbol{\beta}^2\boldsymbol{\beta}^3\boldsymbol{\beta}^4\boldsymbol{\beta}^5\boldsymbol{\beta}^6 \in B'_{n_1}B'_{n_2}B'_{n_3}B_{n_4}B_{n_5}B_{n_6}$, $\boldsymbol{\gamma} = \boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3\boldsymbol{\gamma}^4\boldsymbol{\gamma}^5\boldsymbol{\gamma}^6 \in B''_{n_1}B''_{n_2}B''_{n_3}B_{n_4}B_{n_5}B_{n_6}$). Therefore, $\boldsymbol{\alpha}\{(0)\} + \boldsymbol{\beta}\{(1)\} + \boldsymbol{\gamma}\{(2)\} \neq 0(mod\ 3)$ for every points $\boldsymbol{\alpha} \in P_n\{(0)\}, \boldsymbol{\beta} \in B'_{n_1}B'_{n_2}B_{n_3}\{(1)\}, \boldsymbol{\gamma} \in B''_{n_1}B''_{n_2}B_{n_3}$ $\{(2)\}$ ($\boldsymbol{\alpha} \in P_n\{(0)\}, \boldsymbol{\beta} \in B'_{n_1}B'_{n_2}B'_{n_3}B_{n_4}B_{n_5}B_{n_6}\{(1)\}$, $\boldsymbol{\gamma} \in B''_{n_1}B''_{n_2}B''_{n_3}B_{n_4}B_{n_5}B_{n_6}\{(2)\}$).

Theorem 3: Let P_{n_1}, P_{n_2}, P_{n_3} be b-saturated sets. If two of the sets P_{n_1}, P_{n_2}, P_{n_3}, say P_{n_1}, P_{n_2} are even then $C_{n+1} = P_n\{(0)\} \cup B'_{n_1}B'_{n_2}B_{n_3}\{(1)\} \cup B''_{n_1}B''_{n_2}B_{n_3}\{(2)\}$ is a cap set in $AG(n+1, 3)$, where $P_n = P_{n_1}P_{n_2}B_{n_3} \cup P_{n_1}B_{n_2}P_{n_3} \cup B_{n_1}P_{n_2}P_{n_3}$, $n = \sum_1^3 n_i$ and n_1, n_2, n_3 are any integer numbers.

Theorem 4: Let P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5} and P_{n_6} be b-saturated sets, If three of the sets P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}, P_{n_6}, say P_{n_1}, P_{n_2} and P_{n_3} are even, then $C_{n+1} = P_n\{(0)\} \cup B'_{n_1}B'_{n_2}B'_{n_3}B_{n_4}B_{n_5}B_{n_6}\{(1)\} \cup B''_{n_1}B''_{n_2}B''_{n_3}B_{n_4}B_{n_5}B_{n_6}\{(2)\}$ is a cap set in $AG(n+1, 3)$, where $P_n = \cup_1^{10} A_i$, $n = \sum_1^6 n_i$, $n_1, n_2, n_3, n_4, n_5, n_6$ are any integer numbers and the sets $A_1, A_2, \dots, A_{10}$ are already defined above.

REFERENCES

[1] R. C. Bose, "Mathematical theory of the symmetrical factorial design", *Sankhya*, vol. 8, pp. 107-166, 1947. [Online]. Available: https://www.jstor.org/stable/25047939

[2] J. A. Grochow, "New application of the polynomial methods: The cap set conjecture and beyond", *Bulletin (New Series) Of The American Mathematical Society*, vol. 56, no. 1, pp. 29-64, 2019. [Online]. Available: https://doi.org/10.1090/bull/1648

[3] L. Davis and D. Maclagan, "The card game set", *The Mathematical Intelligencer*, vol. 25(3), pp. 33-40, 2003. [Online]. Available: https://doi.org/10.1007/BF02984846

[4] J. W. P. Hirschfeld and L. Storm, "The packing problem in statistics, coding theory and finite projective spaces", *Journal Statistical Planning and Inference*, no. 72, pp. 355-380, 1998. [Online]. Available: https://doi.org/10.1016/S0378-3758(98)00043-3

[5] Y. Edel and Bierbrauer, "Large caps in small spaeces", *Desgine, Codes and Cryptoraphly*, vol. 23(2), pp. 197-212, 2001. [Online]. Available: https://doi.org/10.1023/a:1011216716700

[6] A. C. Mukhopadhyay, "Lower bounds on $m_t(r, s)$ ", *Journal of Combinatorial Theory*, Seires A, vol. 25(1), pp. 1-13, 1971. [Online]. Available: https://doi.org/10.1016/0097-3165(78)90026-2

[7] B. Qvist, "Some remarks concerning curves of the second degree in a finite plane", *Ann Acad. Sci. Fenn*, Ser. A, vol. 134, p. 27. 1952.

[8] G. Pellegrino, "Sul Massimo ordinedelle calotte in $S_{4,3}$", *Matematiche* (Catania), vol. 25, pp. 1-9, 1970.

[9] R. Hill, "On the largest size of cap in $S_{5,3}$ ", *Atzi Acad.Naz.Likei Rendicondi*, vol. 54, pp. 378-384, 1973.

[10] Y. Edel, S. Ferret, I. Landjev and L. Storme, "The classification of the largest caps in $AG(5,3)$", *Journal of Combinatorial Theory*, ser. A, vol. 99, pp. 95-110, 2002. [Online]. Available: https://doi.org/10.1006/jcta.2002.3261

[11] Y. Edel and J. Bierbrauer, "41 is the largest size of a cap in $PG(4,4)$ ", *Designs, Codes and Cryptography*, vol. 16, pp. 151-160, 1999. [Online]. Available: https://doi.org/10.1023/a:1008389013117

[12] A. Potechin, "Maximal caps in AG (6, 3)", *Designs, Codes and Cryptography*, vol. 46, pp. 243-259, 2008. [Online]. Available: https://doi.org/10.1007/s10623-007-9132-z

[13] A. A. Davidov, G. Faina, S. Marcugini and F. Pambianco, "Computer search in projective planes for the sizes of complete arcs", *J. Geometry*, vol. 82, pp. 50-62, 2005. [Online]. Available: https://doi.org/10.1007/s00022-004-1719-1

[14] A. R. Calderbankand P. C. Fishburn, "Maximal three-independent subsets of $\{0, 1\ 2\}^n$ ", *Designs, Codes and Cryptography,* vol. 4, pp. 203-211, 1994. [Online]. Available: https://doi.org/10.1007/bf01388452

[15] B. Romera-Paredes et al. "Mathematical discoveries from program search with large language models", *Nature.*, 625, (7995), pp. 468-475, 2024. [Online]. Available: https://doi.org/10.1038/s41586-023-06924-6

[16] N. D. Versluis, "On The Cap Set Problem, Upper bounds on maximal cardinalities of caps in dimensions seven to ten", *Delft University of Technology*, pp. 1-52, July 2017.

[17] K. Karapetyan, "Large Caps in Affine Space $AG(n, 3)$", *Proceedings of International Conference Computer Science and Information Technologies*, Yerevan, Armenia, pp. 82-83, 2015. [Online]. Available: https://noad.sci.am/publication/149513

[18] I. A. Karapetyan and K. I. Karapetyan, "Complete Caps in Affine Geometry *AG*(*n*, 3)", *Discrete Mathematical Problems Of Pattern Recognition, Part II*, pp. 74-91, 2024. [Online]. Available: https://doi.org/10.1134/S1054661824010097

[19] K. Karapetyan, "On the complete caps in Galois affine space $AG(n, 3)$", *Proceedings of International Conference Computer Science and Information Technologies*, Yerevan, Armenia, p. 205, 2017. [Online]. Available: https://noad.sci.am/publication/149330