On Classification of Moufang Hyperidentities

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***Abstract*—A binary algebra $(Q;\Sigma)$ is called an l-algebra if there exists an operation $A\in\Sigma$ such that $Q(A)$ is a loop. This work provides a description and classification of Moufang hyperidentities in non-trivial binary algebras with a loop operation.**

***Keywords*—Loop, quasigroup, hyperidentity, Moufang loop, Moufang identity**

I. Introduction

A binary algebra $(Q;\Sigma)$ is said to be an invertible algebra if all operations from Σ are quasigroups.

A binary algebra $(Q;\Sigma)$ is called an l-algebra (q-algebra) if there exists an operation $A\in\Sigma$ such that $Q(A)$ is a loop (a quasigroup).

A binary algebra $(Q;\Sigma)$ is called functionally non-trivial, if $|\Sigma|>1$.

Definition 1: A functional variable $X\in]w_1[\cup]w_2[$ is said to be singular in a hyperidentity $w_1=w_2$ if X occurs just once in it and at least one of the following conditions holds:

a) a subword $w=X(\omega_1,\omega_2)$ involves object variables $x\in[w_1]$ and $y\in[w_2]$, each of which occurs just once in w;

b) a subword $w=X(\omega_1,\omega_2)$ has the form $X(\omega_1,x)$ or $X(x,\omega_2)$ and there exists an object variable $y\in[w]$ different from x and occurring just once in w, where $[w]$ is the set of object variables occurring in w.

Lemma 1: ([1]- [3]) In a non-trivial binary algebra with a quasigroup operation, a hyperidentity involving a singular functional variable cannot be satisfied.

Proof 1: Assume not. Let $\mathcal{Q}=(Q;\Sigma)$ be a functionally non-trivial q-algebra such that Σ contains an invertible operation A, and suppose that $\mathcal{Q}$ satisfies the hyperidentity

$$w_1=w_2$$

having a singular functional variable X. We give X two different values A_1 and A_2 from Σ, and for each such substitution, we give the remaining functional variables the same value A from Σ, where, of course, $Q(A)$ is a quasigroup. This gives two identities:

$$w_1'=w_2',\quad w_1''=w_2''.$$

Suppose, for definiteness, that X occurs in w_1. Then the expressions for w_2' and w_2'' are identical, so that $w_2'=w_2''$ and, thus, $w_1'=w_1''$. All the operations in w_1' and w_1'' are equal to A except for A_1 and A_2. Since $Q(A)$ is a quasigroup, then after all cancellations, we get the identity

$$A_1(\omega_1^0,\omega_2^0)=A_2(\omega_1^0,\omega_2^0).$$

Let $X(\omega_1,\omega_2)$ be a subword satisfying condition a) of Definition 1. Apart from x and y, we give each object variable an arbitrary but fixed value in Q. Then ω_1^0 becomes $\lambda(x)$ and ω_2^0 becomes $mu(y)$, where λ and μ are permutations of Q, since they are products of translations in the quasigroup $Q(A)$. Thus, we have

$$A_1(\lambda x,\mu y)=A_2(\lambda x,\mu y).$$

Since λ and μ are permutations, we deduce from this that $A_1=A_2$, contradicting the choice of these operations. Case b) is dealt with in a similar fashion.

Consider the Moufang identity (m_1):

$$x(y\cdot xz)=(xy\cdot x)z \qquad (m_1).$$

If a non-trivial Moufang hyperidentity holds in a non-trivial l-algebra, then by Lemma 1, each operation symbol must appear at least twice. Therefore, from identity $(m1)$, the following 40 hyperidentities arise:

$$X(x,X(y,Y(x,z)))=Y(X(X(x,y),x),z), \qquad (1)$$
$$X(x,Y(y,X(x,z)))=X(Y(X(x,y),x),z), \qquad (2)$$
$$X(x,Y(y,Z(x,z)))=Z(Y(X(x,y),x),z), \qquad (3)$$
$$Y(x,X(y,X(x,z)))=X(X(Y(x,y),x),z), \qquad (4)$$
$$Y(x,X(y,X(x,z)))=Y(X(X(x,y),x),z), \qquad (5)$$
$$X(x,X(y,Y(x,z)))=X(Y(X(x,y),x),z), \qquad (6)$$
$$X(x,X(y,Y(x,z)))=X(X(Y(x,y),x),z), \qquad (7)$$
$$X(x,Y(y,X(x,z)))=Y(X(X(x,y),x),z), \qquad (8)$$
$$X(x,Y(y,X(x,z)))=X(X(Y(x,y),x),z), \qquad (9)$$
$$Y(x,X(y,X(x,z)))=X(Y(X(x,y),x),z), \qquad (10)$$
$$X(x,Y(y,Z(x,z)))=X(Y(Z(x,y),x),z), \qquad (11)$$
$$X(x,Y(y,Z(x,z)))=X(Z(Y(x,y),x),z), \qquad (12)$$
$$X(x,Y(y,Z(x,z)))=Y(X(Z(x,y),x),z), \qquad (13)$$
$$X(x,Y(y,Z(x,z)))=Y(Z(X(x,y),x),z), \qquad (14)$$

$$X(x, Y(y, Z(x, z))) = Z(X(Y(x, y), x), z), \quad (15)$$

$$X(x, X(y, Y(x, z))) = Y(Y(Y(x, y), x), z), \quad (16)$$

$$X(x, X(y, X(x, z))) = Y(Y(X(x, y), x), z), \quad (17)$$

$$X(x, Y(y, X(x, z))) = Y(Y(Y(x, y), x), z), \quad (18)$$

$$Y(x, X(y, X(x, z))) = Y(Y(Y(x, y), x), z), \quad (19)$$

$$X(x, X(y, X(x, z))) = Y(X(Y(x, y), x), z), \quad (20)$$

$$X(x, X(y, Y(x, z))) = Y(Z(Z(x, y), x), z), \quad (21)$$

$$X(x, X(y, Y(x, z))) = Z(Y(Z(x, y), x), z), \quad (22)$$

$$X(x, X(y, Y(x, z))) = Z(Z(Y(x, y), x), z), \quad (23)$$

$$X(x, Y(y, X(x, z))) = Y(Z(Z(x, y), x), z), \quad (24)$$

$$X(x, Y(y, X(x, z))) = Z(Y(Z(x, y), x), z), \quad (25)$$

$$X(x, Y(y, X(x, z))) = Z(Z(Y(x, y), x), z), \quad (26)$$

$$Y(x, X(y, X(x, z))) = Y(Z(Z(x, y), x), z), \quad (27)$$

$$Y(x, X(y, X(x, z))) = Z(Y(Z(x, y), x), z), \quad (28)$$

$$Y(x, X(y, X(x, z))) = Z(Z(Y(x, y), x), z), \quad (29)$$

$$X(x, X(y, X(x, z))) = Y(Y(Y(x, y), x), z), \quad (30)$$

$$X(x, X(y, Y(x, z))) = X(Y(Y(x, y), x), z), \quad (31)$$

$$X(x, X(y, Y(x, z))) = Y(X(Y(x, y), x), z), \quad (32)$$

$$X(x, X(y, Y(x, z))) = Y(Y(X(x, y), x), z), \quad (33)$$

$$X(x, Y(y, X(x, z))) = X(Y(Y(x, y), x), z), \quad (34)$$

$$Y(x, X(y, X(x, z))) = X(Y(Y(x, y), x), z), \quad (35)$$

$$Y(x, X(y, X(x, z))) = Y(X(Y(x, y), x), z), \quad (36)$$

$$Y(x, X(y, X(x, z))) = Y(Y(X(x, y), x), z), \quad (37)$$

$$X(x, Y(y, X(x, z))) = Y(Y(X(x, y), x), z), \quad (38)$$

$$X(x, Y(y, X(x, z))) = Y(X(Y(x, y), x), z), \quad (39)$$

$$X(x, X(y, X(x, z))) = X(Y(Y(x, y), x), z), \quad (40)$$

II. MAIN RESULTS

Definition 2: An l-algebra $(Q; \Sigma)$ is said to have structure (A) (or (A')), if there exists a Moufang loop $Q(\circ)$ such that each operation $A_i \in \Sigma$ is defined by:

$$A_i(x, y) = x \circ (t_i \circ y)$$
$$(A_i(x, y) = (x \circ t_i) \circ y),$$

where $t_i \in Q$ corresponds to operation A_i.

Definition 3: An l-algebra $(Q; \Sigma)$ is said to have structure (B), if there exists a Moufang loop $Q(\circ)$ such that each operation $A_i \in \Sigma$ is defined by:

$$A_i(x, y) = x \circ (t_i \circ y), \quad t_i \in N(Q(\circ))$$

where $N(Q(\circ))$ is the nucleus of the loop $Q(\circ)$.

Definition 4: An l-algebra $(Q; \Sigma)$ is said to have structure (C), if there exists a Moufang loop $Q(\circ)$ such that each operation $A_i \in \Sigma$ is defined by:

$$A_i(x, y) = (x \circ y) \circ t_i, \quad t_i \in Z(Q(\circ))$$

where $Z(Q(\circ))$ is the center of the loop $Q(\circ)$.

Definition 5: An l-algebra $(Q; \Sigma)$ is said to have structure (D), if there exists a Moufang loop $Q(\circ)$ such that each operation $A_i \in \Sigma$ is defined by:

$$A_i(x, y) = x \circ (t_i \circ y)$$

for some $t_i \in N(Q(\circ)) \cap Z(Q(\circ))$.

Definition 6: An l-algebra $(Q; \Sigma)$ is said to have structure (E), if there exists a Moufang loop $Q(\circ)$ such that each operation $A_i \in \Sigma$ is defined by:

$$A_i(x, y) = x \circ (t_i \circ y)$$

for some $t_i \in N(Q(\circ)) \cap Z(Q(\circ))$, such that $t_i^2 = t_i \circ t_i = e$, where e is the identity of the loop $Q(\circ)$.

Definition 7: An l-algebra $(Q; \Sigma)$ is said to have structure (F) (or (F')), if there exists a Moufang loop $Q(\circ)$ such that each operation $A_i \in \Sigma$ is defined by:

$$A_i(x, y) = x \circ (t_i \circ y)$$
$$(A_i = (x, y) = t_i \circ (x \circ y),$$

for some $t_i \in N(Q(\circ)) \cap Z(Q(\circ))$, such that $t_i^3 = t_i \circ t_i \circ t_i = e$, where e is the identity of the loop $Q(\circ)$.

Remark 1: It is clear that if an l-algebra $(Q; \Sigma)$ has any of the above-defined structures, then all its operations $A_i \in \Sigma$ will be loops with unit t_i^{-1}, where $t_i^{-1} \circ t_i = t_i \circ t_i^{-1} = e$.

Theorem 1: In the class of all non-trivial l-algebras, each non-trivial Moufang hyperidentity defined by the equation: $x(y \cdot xz) = (xy \cdot x)z$, is equivalent (in terms of satisfiability) to one of the following twelve hyperidentities:

$$X(x, X(y, Y(x, z))) = Y(X(X(x, y), x), z), \quad (M_1)$$

$$X(x, Y(y, X(x, z))) = X(Y(X(x, y), x), z), \quad (M_2)$$

$$Y(x, X(y, X(x, z))) = X(X(Y(x, y), x), z), \quad (M_3)$$

$$Y(x, X(y, X(x, z))) = Y(X(X(x, y), x), z), \quad (M_4)$$

$$X(x, X(y, Y(x, z))) = X(Y(X(x, y), x), z), \qquad (M_5)$$

$$X(x, X(y, Y(x, z))) = X(X(Y(x, y), x), z), \qquad (M_6)$$

$$X(x, X(y, Y(x, z))) = Y(Y(Y(x, y), x), z), \qquad (M_7)$$

$$X(x, Y(y, X(x, z))) = Y(Y(Y(x, y), x), z), \qquad (M_8)$$

$$X(x, X(y, X(x, z))) = Y(Y(Y(x, y), x), z), \qquad (M_9)$$

$$X(x, Y(y, X(x, z))) = Y(Y(X(x, y), x), z), \qquad (M_{10})$$

$$X(x, Y(y, X(x, z))) = Y(X(Y(x, y), x), z), \qquad (M_{11})$$

$$X(x, X(y, X(x, z))) = X(Y(Y(x, y), x), z). \qquad (M_{12})$$

From this result, in particular, we obtain the following theorem.

Theorem 2: In the class of all non-trivial invertible algebras with a loop operation, every non-trivial Moufang hyperidentity defined by: $x(y \cdot xz) = (xy \cdot x)z$, is equivalent to one of the hyperidentities: (M_1), (M_2), (M_4), (M_5), (M_7), (M_9), (M_{10}), (M_{11}), (M_{12}).

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