Method for Counting the Number of Interior Points of the Standard Arrangement of the Discrete Torus

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***Abstract*—In this paper, we consider an n-dimensional torus with generating cycles of even length and present a method for calculating the number of interior points of standard arrangements.**

***Keywords*—Discrete torus, standard arrangement, interior point.**

I. INTRODUCTION

<u>Definition 1.</u> For any integers

$$1 \le k_1 \le k_2 \le \cdots \le k_n < \infty,$$

the multivalued n-dimensional torus $T^n_{k_1 k_2 \cdots k_n}$ is defined as the set of vertices:

$$T^n_{k_1 k_2 \cdots k_n} = \\ = \{(x_1, x_2, \cdots, x_n) | -k_i + 1 \le x_i \le k_i, x_i \in Z, 1 \le i \le n\},$$

where two vertices $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in $T^n_{k_1 k_2 \cdots k_n}$ are considered neighbours if they differ in exactly one coordinate i, and either:

- $|x_i - y_i| = 1$, or
- $x_i = -k_i + 1$ and $y_i = k_i$, or vice versa.

The sum and difference of two such vectors x and y are defined componentwise as:

$$x \pm y = (x_1 \pm y_1, x_2 \pm y_2, \cdots, x_n \pm y_n) = \\ = z = (z, z_2, \cdots, z_n),$$

where $-k_i + 1 \le z_i \le k_i$ and $z_i \equiv (x_i \pm y_i)(mod\ 2k_i)$.

Let us define the ***norm of a vertex*** $x = (x_1, x_2, \cdots, x_n)$ as the number $\| x \| = \sum_{i=1}^{n} |x_i|$.

The distance between two vertices $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ is defined as $\rho(x, y) = \| x - y \|$.

The ***sphere*** of radius k centered at a point $x \in T^n_{k_1 k_2 \cdots k_n}$ is defined as the set:

$$S^n(x, k) = \{y \in T^n_{k_1 k_2 \cdots k_n} \,|\, \rho(x, y) \le k\}.$$

The ***shell*** **(**or ***spherical layer)*** of radius k centered at x is defined as the set:

$$O^n(x, k) = \{y \in T^n_{k_1 k_2 \cdots k_n} \,|\, \rho(x, y) = k\}.$$

Let $e_i = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ denote ***the unit vector in the i-th direction***, where $\alpha_i = 1$ and $\alpha_j = 0$ for $j \ne i$. Also, let $\tilde{1} = (1,1, \cdots ,1)$ and $\tilde{0} = (0,0, \cdots ,0)$ denote the **all-ones** and **all-zeros** vectors, respectively.

<u>Definition 2.</u> For a given subset $A \subseteq T^n_{k_1 k_2 \cdots k_n}$ we say that a vertex $x \in A$ is an ***interior point*** of A, if all of its neighbouring vertices also belong to A. We denote by $B(A)$ the set of all interior points of A.

For any vertex $x = (x_1, x_2, \cdots, x_n)$ of $T^n_{k_1 k_2 \cdots k_n}$, we define:

- $|x| = (|x_1|, |x_2|, \cdots, |x_n|)$,
- $\delta(x) = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, where $\alpha_i = 1$ if $x_{n-i+1} > 0$ and $\alpha_i = 0$ if $x_{n-i+1} \le 0$.

For two n-dimensional vectors $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ with nonnegative integer coordinates, we say that x lexicographically precedes y (written $x \prec y$) if there exists an index $r, 1 \le r \le n$, such that $x_i = y_i$ for $1 \le i < r$ and $x_r < y_r$.

Now we define an order on the vertices of the torus $T^n_{k_1 k_2 \cdots k_n}$ as follows:

a vertex x precedes a vertex y (written $x \Leftarrow y$), if and only if one of the following conditions holds:

1. $\| x \| < \| y \|$, or
2. $\| x \| = \| y \|$ and $\delta(y)$ lexicographically precedes $\delta(x)$, or
3. $\| x \| = \| y \|$, $\delta(x) = \delta(y)$, and $|y|$ lexicographically precedes $|x|$.

It is easy to check that the order $\Leftarrow$ defined on the vertices of the torus $T^n_{k_1 k_2 \cdots k_n}$ is a linear order.

<u>Definition 3.</u> The first a vertices of the torus $T^n_{k_1 k_2 \cdots k_n}$ taken according to the above defined linear order, are called the ***<u>standard arrangement</u>*** of cardinality a, $0 \le a \le |T^n_{k_1 k_2 \cdots k_n}|$.

The torus $T^n_{k_1 k_2 \cdots k_n}$ with $k_1 = k_2 = \cdots = k_n = 1$ is called the n - ***dimensional unit cube***, and is denoted by E^n.

For a Boolean vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, the set

$$\alpha(T^n_{k_1 k_2 \cdots k_n}) = \{x \in T^n_{k_1 k_2 \cdots k_n} \,|\, \delta(x) = \alpha\}$$

is called the α - ***part*** of the torus $T^n_{k_1 k_2 \cdots k_n}$. It is clear that

$$T^n_{k_1 k_2 \cdots k_n} = \bigcup_{\alpha \in E^n} \alpha(T^n_{k_1 k_2 \cdots k_n})$$

and all α -parts of the torus are isomorphic.

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Moreover, the α - parts of $T^n_{k_1k_2\cdots k_n}$ are arranged according to the order $\Leftarrow$.

For two vertices $x=(x_1,x_2,\cdots,x_n)$ and $y=(y_1,y_2,\cdots,y_n)$, belonging to $\alpha(T^n_{k_1k_2\cdots k_n})$, we define their sum as:

$$x+y=(x_1+y_1,x_2+y_2,\cdots,x_n+y_n)=(z_1,z_2,\cdots,z_n),$$

where $(x_i+y_i)\equiv z_i(mod\ k_i)$, $1\le z_i\le k_i$ when $\alpha_i=1$, and $-k_i+1\le z_i\le 0$ when $\alpha_i=0$, for all $i, 1\le i\le n$.

Let us define the **sphere** (respectively, **shell**) in $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$–part centered at a point $x\in\alpha(T^n_{k_1k_2\cdots k_n})$ with radius k, as follows:

$$S^n_\alpha(x,k)=\left\{y=x+\sum_{i=1}^{n}(-1)^{1+\alpha_i}\cdot r_ie_i \mid \sum_{i=1}^{n}r_i\le k\right\},$$

(respectively, $O^n_\alpha(x,k)=S^n_\alpha(x,k)\setminus S^n_\alpha(x,k-1)$), where r_i are non-negative integers for all $i, 1\le i\le n$.

The subset of internal vertices of a set $A\subseteq\alpha(T^n_{k_1k_2\cdots k_n})$ in the α – part is defined as:

$$B_\alpha(A)=\{x\in A \mid S^n_\alpha(x,1)\subseteq A\}.$$

The linear order $\Leftarrow$ defined above between the vertices of $T^n_{k_1k_2\cdots k_n}$ within each α-part coincides with the diagonal sequence defined in [1], the initial segment of which we again call the standard arrangement.

In the works [2-3], some properties of the standard arrangements of the discrete torus $\boldsymbol{T^n_{k_1k_2\cdots k_n}}$ are proved. In particular, it is proved that the standard arrangement of any cardinality has the maximum number of internal vertices. This work presents a method for calculating the number of internal vertices of the standard arrangement.

II. Counting the Number of Interior Points of the Standard Arrangement

The special structure of standard arrangements allows us to determine the values of $|B(A)|$ and $|B_\alpha(D)|$, where A is the standard arrangement of the torus $T^n_{k_1k_2\cdots k_n}$, and D is the standard arrangement of the α - part $\alpha(T^n_{k_1k_2\cdots k_n})$.

Let us denote $|O^n(x,k)|=F^k_n(k_1,k_2,\cdots,k_n)$ and

$$|O^n_\alpha(x,k)|=f^k_n(k_1,k_2,\cdots,k_n).$$

In particular, $f^k_n(k_1,k_2,\cdots,k_n)=0$ if k does not satisfy the condition

$$0\le k\le\sum_{i=1}^{n}(k_i-1)=K.$$

It is clear that

$$F^k_n(k_1,k_2,\cdots,k_n)=\sum_{\alpha\in E^n}f^{k-\|\alpha\|}_n(k_1,k_2,\cdots,k_n).$$

The numbers $f^k_n(k_1,k_2,\cdots,k_n)$, for $0\le k\le K$, are determined by the following identity:

$$(1+x+x^2+\cdots+x^{k_1-1})(1+x+x^2+\cdots+x^{k_2-1})\cdots(1+x+x^2+\cdots+x^{k_n-1})=$$
$$=\sum_{k=0}^{K}f^k_n(k_1,k_2,\cdots,k_n)\cdot x^k.$$

Let D be the standard arrangement in α– part of cardinality b, $0\le b\le|\alpha(T^n_{k_1k_2\cdots k_n})|=\prod_{i=1}^n k_i$, and A be the standard arrangement in $T^n_{k_1k_2\cdots k_n}$ of cardinality a,

$$0\le a\le|T^n_{k_1k_2\cdots k_n}|=2^n\cdot\prod_{i=1}^{n}k_i.$$

Then the numbers b and a can be represented as:

$$\left.\begin{aligned} b=\sum_{i=0}^{k}f^i_n(k_1,k_2,\cdots,k_n)+b_1,\text{where}\\ 0\le b_1<f^{k+1}_n(k_1,k_2,\cdots,k_n)\end{aligned}\right\},\qquad(1)$$

$$\left.\begin{aligned} a=\sum_{i=0}^{t}F^i_n(k_1,k_2,\cdots,k_n)+a_1,\text{where}\\ 0\le a_1<F^{t+1}_n(k_1,k_2,\cdots,k_n)\end{aligned}\right\}.\qquad(2)$$

Lemma 1. If $0\le b_1<f^{k+1}_n(k_1,k_2,\cdots,k_n)$, then b_1 can be uniquely represented in the following form:

$$b_1=\sum_{r=1}^{\mu}\sum_{j=0}^{l_r}f^{k(r,j)}_{n-n_r}(k_{n_r+1},k_{n_r+2},\cdots,k_n),\qquad(3)$$

where $1\le n_1<n_2<\cdots<n_\mu<n$, $l_0=0$, $0\le l_r<k_{n_r}-1$ for $1\le r\le\mu$ and

$$k(r,j)=k-\sum_{i=1}^{n_r}k_i+\sum_{i=0}^{r-1}l_i+n_r+r+j.$$

Proof. Indeed, for b_1 there exists such a smallest integer n_1, $1\le n_1<n$, for which

$$f^{k(1,0)}_{n-n_1}(k_{n_1+1},k_{n_1+2},\cdots,k_n)\le b_1<$$
$$<\sum_{j=0}^{k_{n_1}-1}f^{k(1,j)}_{n-n_1}(k_{n_1+1},k_{n_1+2},\cdots,k_n).\qquad(4)$$

Therefore,

$$b_1=\sum_{j=0}^{l_1}f^{k(1,j)}_{n-n_1}(k_{n_1+1},k_{n_1+2},\cdots,k_n)+b_2,\text{where}$$
$$0\le b_2<f^{k(1,l_1)+1}_{n-n_1}(k_{n_1+1},k_{n_1+2},\cdots,k_n).$$

It is evident that, by analogy with (4), one can find the smallest number n_2, $n_1<n_2<n$, for which

$$f^{k(2,0)}_{n-n_2}(k_{n_2+1},k_{n_2+2},\cdots,k_n)\le b_2<$$
$$<\sum_{j=0}^{k_{n_2}-1}f^{k(2,j)}_{n-n_2}(k_{n_2+1},k_{n_2+2},\cdots,k_n)$$

and thus,

$$b_2=\sum_{j=0}^{l_2}f^{k(2,j)}_{n-n_2}(k_{n_2+1},k_{n_2+2},\cdots,k_n)+b_3,$$

where $0\le b_3<f^{k(2,l_2)+1}_{n-n_2}(k_{n_2+1},k_{n_2+2},\cdots,k_n)$,

and so on, which completes the proof of the lemma. □

According to (3), the number $p_k(b_1)$ of all interior vertices of the standard arrangement D, belonging to the shell $O_\alpha^n(\alpha, k)$, is determined by the following formula:

$$p_k(b_1) = \sum_{r=1}^{\mu} \sum_{j=0}^{l_r} f_{n-n_r}^{k(r,j)-1} \left(k_{n_r+1}, k_{n_r+2}, \cdots, k_n\right). \quad (5)$$

Lemma 2. If A is the standard arrangement in $T_{k_1 k_2 \cdots k_n}^n$ of the cardinality given by (2), where

$$0 \le a_1 < F_n^{t+1}(k_1, k_2, \cdots, k_n),$$

then the number a_1 can be uniquely represented in the following form:

$$\left. \begin{array}{c} a_1 = \sum_{i=1}^{s} \sum_{\alpha \in E^{n-m_i}} f_n^{t-\|\alpha\|-m_i+i}(k_1, k_2, \cdots, k_n) + a_0, \\ where\ 1 \le m_1 < m_2 < \cdots < m_s \le n\ and \\ 0 \le a_0 < f_n^{t-n+s}(k_1, k_2, \cdots, k_n) \end{array} \right\}. \quad (6)$$

It is easy to see that when $k_1 = k_2 = \cdots = k_n = 1$, formula (6) coincides with representation (3.1) of work [4].

According to (6) and (5), the number $P_t(a_1)$ of all interior vertices of the standard arrangement A, belonging to the shell $O^n(\tilde{0}, t)$, is determined by the following formula:

$$\left. \begin{array}{c} P_t(a_1) = \sum_{i=1}^{s} \sum_{\alpha \in E^{n-m_i}} f_n^{t-\|\alpha\|-m_i+i-1}(k_1, k_2, \cdots, k_n) + \\ +p_{t-n+s-1}(a_0) \end{array} \right\}. \quad (7)$$

The following theorem follows from (5) and (7).

Theorem. If D is the standard arrangement in the α–part of cardinality (1), and A is the standard arrangement in $T_{k_1 k_2 \cdots k_n}^n$ of cardinality (2), then

$$|B_\alpha(D)| = \sum_{i=0}^{k-1} f_n^i(k_1, k_2, \cdots, k_n) + p_k(b_1),$$

$$|B(A)| = \sum_{i=0}^{t-1} F_n^i(k_1, k_2, \cdots, k_n) + P_t(a_1).$$

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