Elements of an Analysis of the Category with Computable Functions as Arrows

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***Abstract*—Let N be the set of nonnegative integers. The article defines a certain structure with computable subsets of N as objects and partial computable functions (having computable domain) as arrows. It is proved that this structure is a category (namely, Npcomp). It is shown in what capacity the notions of "co-product of two objects", "product of two objects", "cone for given diagram", and "limit of given diagram" are presented in the category Npcomp. In particular, it is proved that the co-product of two objects in this category is the joint of these two objects.**

***Keywords*—Category, partial computable functions, computable sets.**

I. Introduction

We will use the terminology of [1] and [2]. The notions used in category theory and in this article can be found in the works of [3],[4],[5].

In particular, we recall the following notations and definitions:

Let N be the set of nonnegative integers.
Let $\mathcal{N} = \{A \subseteq N\}$.

$$\tau(x,y) = \frac{1}{2}(x^2 + 2xy + y^2 + 3x + y)$$

τ is a computable one-one mapping of $N \times N$ onto N.

Let π_1 and π_2 denote the inverse functions $\pi_1(\tau(x,y)) = x$ and $\pi_2(\tau(x,y)) = y$.

$f \restriction x$ denotes the restriction of f to arguments $y < x$, $f \restriction A$ denotes the restriction of f to arguments $y \in A$. The identity function is characterised by the rule $f(x) = x$. Each set A has its own identity function, called the *identity function on* A, denoted id_A, the domain of which is the set A. Thus the image of id_A is also A, i.e., $\mathrm{id}_A\colon A \longrightarrow A$. On the set-theoretic account, $\mathrm{id}_A = \{\langle x,x\rangle\colon x \in A\}$ (see [3] §2.1).

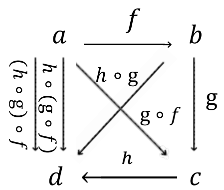
Let A *join* B, written $A \oplus$ B, be $\{2x\colon x \in A\} \cup \{2x+1\colon x \in B\}$.

Axiomatic definition of a category. A *category* $\mathcal{E}$ comprises

1) a collection of things called $\mathcal{E}$-*objects*;
2) a collection of things called $\mathcal{E}$-*arrows*;
3) operations assigning to each $\mathcal{E}$-arrow f an $\mathcal{E}$-object $\mathrm{dom}\, f$ (the "domain" of f) and an $\mathcal{E}$-object $\mathrm{cod}\, f$ (the "codomain" of f). If $a = \mathrm{dom}\, f$ and $b = \mathrm{cod}\, f$ we display this as
$$f : a \to b \text{ or } a \xrightarrow{f} b;$$
4) an operation assigning to each pair $\langle \mathrm{g}, f\rangle$ of $\mathcal{E}$-arrows with $\mathrm{dom}\, \mathrm{g} = \mathrm{cod}\, f$, an $\mathcal{E}$-arrow $\mathrm{g} \circ f$, *the composite of* f *and* g, having $\mathrm{dom}(\mathrm{g} \circ f) = \mathrm{dom}\, f$ and $\mathrm{cod}(\mathrm{g} \circ f) = \mathrm{cod}\, \mathrm{g}$, i.e. $\mathrm{g} \circ f\colon \mathrm{dom}\, f \to \mathrm{cod}\, \mathrm{g}$, and such that the following condition is obtained:

Associative Law: Given the configuration
$$a \xrightarrow{f} b \xrightarrow{\mathrm{g}} c \xrightarrow{h} d$$
of $\mathcal{E}$-objects and $\mathcal{E}$-arrows, then $h \circ (\mathrm{g} \circ f) = (h \circ \mathrm{g}) \circ f$.

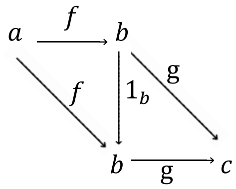
The associative law asserts that a diagram having the form



always commutes;

5) an assignment to each $\mathcal{E}$-object b of an $\mathcal{E}$-arrow $1_b : b \to b$, called *the identity arrow on b,* such that

Identity Law: For any $\mathcal{E}$-arrows $f : a \to b$ and $\mathrm{g} : b \to c$, $1_b \circ f = f$, and $\mathrm{g} \circ 1_b = \mathrm{g}$, i.e., the diagram



commutes.

A Set denotes a category in which the objects are all sets and the arrows are all functions between sets (see [3] §2.3).

II. PRELIMINARIES

A set function $f : A \to B$ is said to be *injective*, or *one-one* when no two distinct inputs give the same output, i.e. for inputs $x, y \in A$, if $f(x) = f(y)$, then $x = y$.

A set function $f : A \to B$ is *onto*, or *surjective,* if the codomain B is the range of f, i.e., for each $y \in$ B there is some $x \in A$ such that $y = f(x)$, i.e., every member of B is an output for f.

A function that is both injective and subjective is called *bijective*.

A function that is related to f as follows:

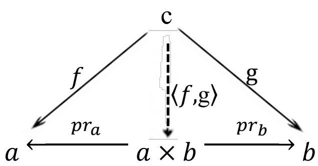
$$g \circ f = id_A \text{ and } f \circ g = id_B$$

is said to be an *inverse of* f. This is an essentially arrow-theoretic idea, and leads to a new definition.

An Ɛ-arrow $f : a \to b$ is *iso*, or *invertible*, in Ɛ if there is an Ɛ-arrow $g : b \to a$, such that $g \circ f = 1_a$ and $f \circ g = 1_b$.

Objects a and b are *isomorphic* in Ɛ, denoted $a \cong b$, if there is an Ɛ-arrow $f : a \to b$ that is iso in Ɛ, i.e., $f : a \cong b$.

Definition. A *product* in a category Ɛ of two objects a and b is an Ɛ-object $a \times b$ together with a pair $(pr_a: a \times b \to a,\ pr_b: a \times b \to b)$ of Ɛ-arrows such that for *any* pair of Ɛ-arrows of the form $(f : c \to a,\ g : c \to b)$, there is exactly one arrow $\langle f, g\rangle: c \to a \times b$ making



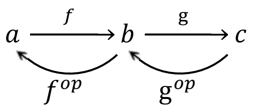
commute, i.e., such that $pr_a \circ \langle f, g\rangle = f$ and $pr_b \circ \langle f, g\rangle = g$. $\langle f, g\rangle$ is the *product arrow of* f *and* g *with* respect to the *projections* pr_a, pr_b.

Notice that product of a and b *is only defined up to isomorphism* (see [3] §3.8).

If Σ is a statement in the basic language of categories, the *dual* of Σ, Σ^{op}, is the statement obtained by replacing "dom" by "cod", "cod" by "dom" and "$h = g \circ f$" by "$h = f \circ g$". Thus, all arrows and composites referred to by Σ are reversed in Σ^{op}. The notion or construction described by Σ^{op} is said to be *dual* to that described by Σ .

From a given category Ɛ , we construct its *dual* or *opposite* category $Ɛ^{op}$ as follows:

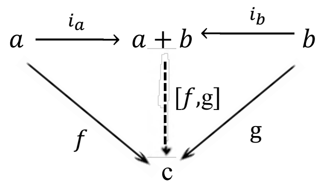
Ɛ and $Ɛ^{op}$ have the same objects. For each Ɛ-arrow $f : a \to b$ we introduced an arrow $f^{op}: b \to a$ in $Ɛ^{op}$, these being all and only the arrows in $Ɛ^{op}$. The composite $f^{op} \circ g^{op}$ is defined precisely when $g \circ f$ is defined in Ɛ and has



$f^{op} \circ g^{op} = (g \circ f)^{op}$. Note that dom $(f^{op}) =$ cod f and cod$(f^{op}) =$ dom f.

The dual notion to "product" is the *co-product* or *sum* of objects, which, by the duality principle we directly define as follows.

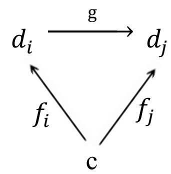
Definition. A *co-product* of Ɛ-objects a and b is an Ɛ- object $a + b$ together with a pair $(i_a: a \to a + b,\ i_b: b \to a + b)$ of Ɛ-arrows such that for any pair of Ɛ-arrows of the form $(f : a \to c,\ g : b \to c)$, there is exactly one arrow $[f, g]: a + b \to c$ making



commute, i.e. such that $[f, g] \circ i_a = f$ and $[f, g] \circ i_b = g$.

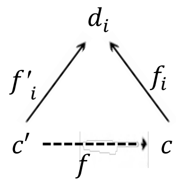
$[f, g]$ is called the *co-product* arrow of f and g with respect to the *injections* i_a and i_b (see [3] §3.9).

A *cone* for diagram D consists of an Ɛ-object c together with an Ɛ-arrow $f_i: c \to d_i$ for each object d_i in D, such that



commutes whenever g is an arrow in the diagram D. We use the symbolism $\{f_i: c \to d_i\}$ to denote a cone for D.

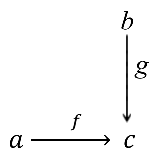
A *limit* for a diagram D is a D-cone $\{f_i: c \to d_i\}$ with the property that for any other D-cone $\{f'_i : c' \to d_i\}$, there is exactly one arrow $f: c' \to c$ such



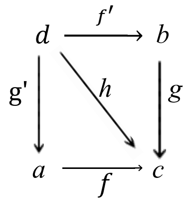
commutes for every object d_i in D.

This limiting cone, when it exists, is said to have the *universal property* with respect to D-cones. It is universal amongst such cones – any other D-cone factors uniquely through it as the last diagram. A limit for diagram D is unique up to isomorphism: - if $\{f_i: c \to d_i\}$ and $\{f'_i: c' \to d_i\}$ are both limits of D, then the unique commuting arrow $f: c' \dashrightarrow c$ above is iso (its inverse is the unique commuting arrow $c \dashrightarrow c'$ whose existence follows from the fact that $\{f'_i: c' \to d_i\}$ is a limit) (see [3] §3.11).

Definition. A *pullback* of a pair $a \xrightarrow{f} c \xleftarrow{g} b$ of Ɛ-arrows with a common codomain is a limit in Ɛ for the diagram

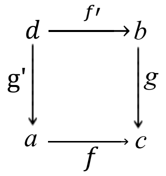


A cone for this diagram consists of three arrows f', h, g', such that



commutes. But this requires that $h = g \circ f' = f \circ g'$.

So, it can be said that a cone is a pair $a \xleftarrow{g'} d \xrightarrow{f'} b$ of $\mathcal{E}$-arrows such that the "square"

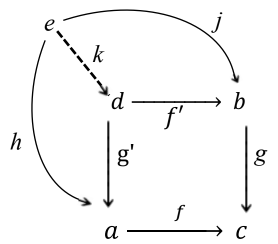


commutes, i.e., $f \circ g' = g \circ f'$.

Thus, we have, by the definition of universal cone, that a *pullback* of the pair $a \xrightarrow{f} c \xleftarrow{g} b$ in $\mathcal{E}$ is a pair of $\mathcal{E}$-arrows pair $a \xleftarrow{g'} d \xrightarrow{f'} b$ such that

(i) $f \circ g' = g \circ f'$, and

(ii) whenever $a \xleftarrow{h} e \xrightarrow{j} b$ are such that $f \circ h = g \circ j$, then

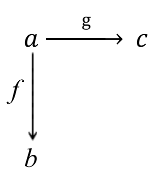


there is exactly one $\mathcal{E}$-arrow $k : e \dashrightarrow d$ such that $h = g' \circ k$ and $j = f' \circ k$.

The inner square (f, g, f', g') of the diagram is called a *pullback square,* or *Cartesian square.* We also say that f' arises by *pulling back f along* g, and g' arises by *pulling back* g *along f* (see [3] §3.13).

The dual of a pullback of a pair of arrows with a common codomain is a *pushout* of the two arrows with a common domain:

a pushout of $b \xleftarrow{f} a \xrightarrow{g} c$ is a pushout for the diagram



In **Set** it is obtained by forming the disjoint union $b + c$ and then identifying $f(x)$ with $g(x)$, for each $x \in a$ (see [3] §3.14).

III. Results

Definition of *Npcomp* category

The *objects* are computable subsets of N (i.e., computable elements of $\mathcal{N}$).

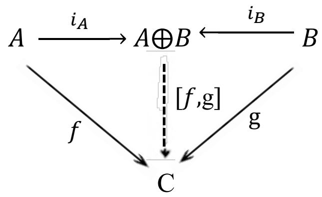
The *arrows* are partial computable functions, the domains of which are computable sets.

Composition. With the above definition of objects and arrows, the associative law is satisfied, i.e., if $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Identity arrow. The *identity arrow* 1_A on the object A of the category $Npcomp$ is defined to be the identity function on A. The above definition of arrows allows us to assert that the identity function satisfies the identity law.

Proposition 1: In category $Npcomp$, the co-product of A and B is their joint, $A \oplus B$.

Proof: Let $i_A(x) = 2x, i_B(x) = 2x + 1$.



Suppose we are given some other set C with a pair of maps $f: A \to C$, $g: B \to C$. Then we define $[f,g]$: by the rule

$$[f,g](x) = \begin{cases} f(m), & \text{if } (\exists m)(x = 2m) \\ g(m), & \text{if } (\exists m)(x = 2m + 1). \end{cases}$$

Then we have $[f,g] \circ i_A(x) = f(x)$ for all $x \in A$ and $[f,g] \circ i_B(x) = f(x)$ for all $x \in B$ and so $[f,g] \circ i_A = f$ and the above diagram commutes.

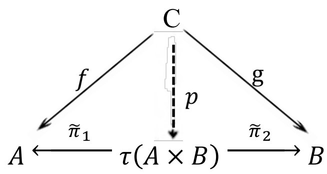
Now we will show that $[f,g]$ is the only arrow that can make the diagram commute. For if $f(x) = z$ and $g(x) = y$, then since $[f,g] \circ i_A = f$, must be $[f,g] \circ i_A(x) = f(x)$, i.e. $z = f(x)$. Similarly, if $[f,g] \circ i_B = g$ we must have $y = g(x)$.

Besides, since the functions f, g, i_A, i_B are partial computable functions with computable domains, the function $[f,g]$ defined in the above way is also a partial computable function with a computable domain.□

Proposition 2: In category $Npcomp$, the product of A and B is the set $\tau(A \times B) = \{\tau(x, y): x \in A, y \in B\}$.

Proof: Given sets A and B and the functions π_1 and π_2 presented in the Introduction generate maps $\tilde{\pi}_1: \tau(A \times B) \to A$ and $\tilde{\pi}_2: \tau(A \times B) \to B$ (where $\tilde{\pi}_1 = \pi_1 \restriction \tau(A \times B)$ and $\tilde{\pi}_2 = \pi_2 \restriction \tau(A \times B)$).

So $\tilde{\pi}_1(\tau(x, y)) = x$, $\tilde{\pi}_2(\tau(x, y)) = y$.



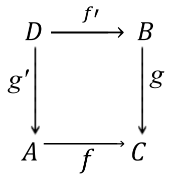
Suppose we are given some other set C with a pair of maps $f: C \to A$, $g: C \to B$. Then we define $p: C \to \tau(A \times B)$ by the rule $p(x) = \tau(f(x), g(x))$.

Then we have $\tilde{\pi}_1(p(x)) = f(x)$ and $\tilde{\pi}_2(p(x)) = g(x)$, for all $x \in C$, so $\tilde{\pi}_1 \circ p = f$ and $\tilde{\pi}_2 \circ p = g$. Thus, the above diagram commutes.

The uniqueness of arrow p is proved approximately the same way as in Proposition 1.

In addition, since the functions f, g, $\tilde{\pi}_1, \tilde{\pi}_2$ are partial computable functions with computable domains, the function p defined in the above way is also a partial computable function with a computable domain.□

Theorem 1: In category $Npcomp$, the pullback



of two arrows f and g is defined by putting

$$D = \{\tau(x, y) \in A, y \in B, \text{ and } f(x) = g(y)\}$$

with f' and g' as the projections:

$$f'(\tau(x, y)) = y$$

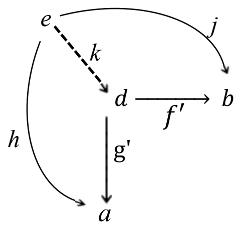
$$g'(\tau(x, y)) = x.$$

Proof: As can be seen from the definition of functions f', g' (in the formulation of the theorem)

$$f \circ g' = g \circ f'.$$

Now, suppose we are given some other set e with a pair of maps $h : e \to a$, $j : e \to b$. Then we define k $: e \to d$ by the rule $k(x) = \tau(h(x), j(x))$.

Let us prove that the outer square commutes.



Then we have $f'(k(x)) = j(x)$, and $g'(k(x)) = h(x)$ for all $x \in e$, so $f' \circ k = j$, $g' \circ k = h$ (remind that in $Npcomp$ category, e, d, a, b are sets).

Function k is exactly one for which the last diagram commutes. Indeed, if $k(x) = \tau(y, x)$, then since $f' \circ k = j$, then $f'(k(x)) = j(x)$, i.e., $y = j(x)$. Similarly, if $g' \circ k = h$, then $z = h(x)$.

Moreover, since the functions h, j, f', g' are partial computable functions with computable domains, the function k defined in the above way is also a partial computable function with a computable domain.□

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