Study of nearly KK-MBF-type functions

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***Abstract* — Recently, a special class of monotone Boolean functions (MBFs), known as the Kruskal-Katona class, arising from the theory of shadow minimization of finite set systems, was studied, and the deadlock-resolving set for these functions was determined. The zeros of KK-MBFs correspond to initial segments of the lexicographic order on layers of the binary cube; consequently, the units correspond to initial segments of the reverse-lexicographic order. In this paper, we extend the KK-MBF class by allowing a single unit (or a single zero) to appear at a specific location within the initial segment of the lexicographic (or reverse-lexicographic) order on the layers corresponding to the zeros (or units) of the function.**

***Keywords*—Monotone Boolean functions, nearly KK-MBF class, deadlock-resolving set, recognition.**

I. INTRODUCTION

The class of KK-MBF-type functions, originating from the theory of shadow minimization of finite set systems, was studied in [1] in the context of query-based function recognition. The zeros of KK-MBF functions correspond to initial segments of the lexicographic order on layers of the binary cube; consequently, the units of these functions correspond to initial segments of the reverse-lexicographic order. The unique deadlock-resolving set for KK-MBFs, along with estimates of its cardinality, was identified in [1].

In this paper, we extend the KK-MBF class to what we call *Nearly KK-MBF*, by allowing a single unit (or zero) to appear at a specific location within the initial segment of the lexicographic (or reverse-lexicographic) order on the layers, corresponding to the zeros (or units) of the function. We determine a deadlock-resolving set for this extended class, and provide estimates of its cardinality.

The rest of the paper is organized as follows. Section 2 provides the necessary definitions and preliminaries. In Section 3, we present the KK-MBF class and introduce key concepts such as corner points. Section 4 introduces nearly KK-MBF-type functions and investigates the cardinality estimates of their deadlock-resolving sets. The paper ends with concluding remarks in Section 5.

II. PRELIMINARIES

Let $B^n = \{(x_1, \cdots, x_n) \mid x_i \in \{0,1\}, i = 1, \cdots, n\}$ denote the set of vertices of the n-dimensional binary cube. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ be two vertices of B^n. α *precedes* β (in the *coordinate-wise order*), denoted as $\alpha \preccurlyeq \beta$, if and only if $\alpha_i \le \beta_i$ for $1 \le i \le n$. α and β are *comparable* if $\alpha \preccurlyeq \beta$ or $\beta \preccurlyeq \alpha$, otherwise, they are *incomparable*. A set of pairwise incomparable vertices in B^n is also called a *Sperner* family.

We will also use the *lexicographic order* of vertices. α *precedes* β lexicographically, denoted $\alpha \preccurlyeq_{lex} \beta$, if either there exists an integer k, $1 \le k \le n$, such that $\alpha_k < \beta_k$ and $\alpha_i = \beta_i$ for $i < k$, or $\alpha = \beta$.
Let $L_k = \{(\alpha_1, \cdots, \alpha_n) \in B^n \mid \sum_{i=1}^{n} \alpha_i = k\}$. We call L_k the k-th layer of B^n.

Let $\mathcal{M}$ be a set of vertices from the k-th layer. The *lower* shadow of $\mathcal{M}$, denoted $\delta^{k-1}\mathcal{M}$, is the set of vertices from the $(k-1)$-th layer that are less than some vertex in $\mathcal{M}$ (with respect to the coordinate-wise order). Similarly, the *upper shadow* of $\mathcal{M}$, denoted $\delta^{k+1}\mathcal{M}$, is the set of vertices from the $(k+1)$-th layer that are greater than some vertex of $\mathcal{M}$.

A Boolean function $f: B^n \to \{0,1\}$ is *monotone* if, for every pair of vertices $\alpha, \beta \in B^n$, $\alpha \preccurlyeq \beta$ implies $f(\alpha) \le f(\beta)$. The vertices of B^n where f takes the value "1" are called *units* of the function; the vertices, where f takes the value "0", are called *zeros*. α^1 is a *lower unit* of the function if $f(\alpha^1) = 1$, and $f(\alpha) = 0$ for every $\alpha \in B^n$, such that $\alpha \prec \alpha^1$. α^0 is an *upper zero* of the function if $f(\alpha^0) = 0$, and $f(\alpha) = 1$ for every $\alpha \in B^n$ such that $\alpha^0 \prec \alpha$. $\min T(f)$ and $\max F(f)$ denote the sets of lower units and upper zeros, respectively. Obviously, both $\min T(f)$ and $\max F(f)$ are Sperner families in B^n.

III. KK-MBF CLASS

Definition 1 [1]. A monotone Boolean function f belongs to the KK-MBF class if, for any $\alpha \in L_k$ $(1 \le k \le n)$, $f(\alpha) = 0$ implies $f(\beta) = 0$ for all $\beta \in L_k$ such that $\beta \prec_{lex} \alpha$. This also implies that if $f(\alpha) = 1$, then $f(\beta) = 1$ for all β from L_k such that $\beta \succ_{lex} \alpha$.

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Definition 2 [1]. A zero vertex α of a KK-MBF-type function f is called a *0-corner point* if:

(1) $f(\beta)=1$ for all β from the same layer such that $\beta \succ_{lex} \alpha$, and

(2) $f(\beta)=1$ for all $\beta, \beta \succ \alpha$ (component-wise order).

Similarly, a unit vertex α of a KK-MBF-type function f is called a *1-corner point* if:

(1) $f(\beta)=0$ for all β from the same layer such that $\beta \prec_{lex} \alpha$, and

(2) $f(\beta)=0$ for all $\beta, \beta \prec \alpha$ (component-wise order).

Thus, the 1-corner points form a subset in $\min T(f)$ and are the lexicographically smallest lower units on the layers; the 0-corner points form a subset in $\max F(f)$ and are the lexicographically greatest upper zeros on the layers.

A layer can contain either one corner point (either a 1-corner or a 0-corner), or two corner points (both a 1-corner and a 0-corner). If a layer contains both, the 1-corner point is the lexicographically right neighbor of the 0-corner point. An example of a KK-MBF function on B^5 is given in Figure 1. Uncolored vertices represent the zeros of the function, blue vertices represent the units, and stars indicate the corner points.

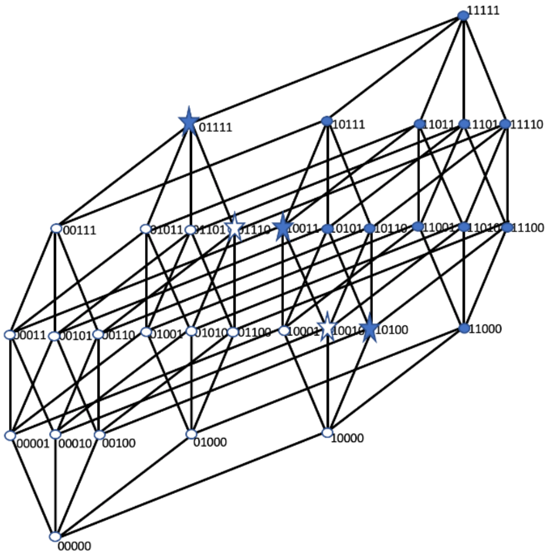

Figure 1.
An example of a KK-MBF function on B^5

Definition 3 [1]. A set of vertices $G(f,\mathcal{S}) \subseteq B^n$ is called a *resolving set* for the pair $(f,\mathcal{S})$ if the following:

a) a function g belongs to the class $\mathcal{S}$, and

b) $g(\alpha)=f(\alpha)$ for all $\alpha \in G(f,\mathcal{S})$

together imply that $g=f$.

A resolving set $G(f,\mathcal{S})$ is called a *deadlock resolving set* for $(f,\mathcal{S})$, if no proper subset of it is resolving for the pair $(f,\mathcal{S})$.

It was proved in [1] that the unique deadlock resolving set for the class the KK-MBF is the union of the sets of its 1-corner and 0-corner points.

IV. NEARLY KK-MBF CLASS

We first define the lexicographic distance between vertices of the same layer of B^n.

Let $\alpha, \beta \in L_k$, and $\alpha \prec_{lex} \beta$. If there is no $\gamma \in L_k$ such that $\alpha \prec_{lex} \gamma \prec_{lex} \beta$, then the *lexicographic distance* between α and β, denoted $lex\_dist(\alpha,\beta)$, is 1. In this case, α and β are lexicographically neighbors.

In general, the lexicographic distance between α and β is $r, r \geq 2$ if there exist vertices $\alpha_1, \alpha_2, \alpha_1 \cdots, \alpha_{r-1}$ from L_k such that:

$$\alpha \prec_{lex} \alpha_1 \prec_{lex} \alpha_2 \prec_{lex} \cdots \prec_{lex} \alpha_{r-1} \prec_{lex} \beta, \text{ and}$$
$$lex\_dist(\alpha,\alpha_1) = lex\_dist(\alpha_1,\alpha_2) = \cdots le.x\_dist(\alpha_{r-1},\beta) = 1$$

Definition 3. A monotone Boolean function f is *nearly* KK-MBF if, for any $\alpha \in L_k$ $(1 \leq k \leq n)$, the following holds:
if $f(\alpha)=0$, then $f(\beta)=0$ for all β from L_k such that $\beta \prec_{lex} \alpha$ and $lex\_dist(\alpha,\beta) \geq 2$.
This also implies that if $f(\alpha)=1$, then $f(\beta)=1$ for all β such that $\beta \succ_{lex} \alpha$ and $lex\_dist(\beta,\alpha) \geq 2$.

For a KK-MBF, each layer of the cube exhibits the pattern shown in Figure 2, where 0s indicate zero-vertices and 1s indicate unit vertices. Each of the segments labeled (1)-(2) may be empty. In contrast, a nearly KK-MBF allows a unit to appear between zeros, or a zero to appear between units. Consequently, the structure at each layer resembles the configuration shown in Figure 3, where each of the parts (1)-(4) may be empty.

$$\overbrace{0 \cdots \mathbf{0}}^{(1)}\ \overbrace{\mathbf{1} \cdots 1}^{(2)}$$

Figure 2.
The structure of each layer for a KK-MBF

$$\overbrace{0 \cdots 0}^{(1)}\ \overbrace{\mathbf{1}}^{(2)}\ \overbrace{\mathbf{0}}^{(3)}\ \overbrace{1 \cdots 1}^{(4)}$$

Figure 3.
The structure of each layer for a nearly KK-MBF

Part (1) cannot contain units, as all vertices in this segment are at a lexicographic distance of at least 2 from the zero-vertex in part (3). Similarly, part (4) cannot contain zeros, since all vertices are at a lexicographic distance of at least 2 from the unit vertex in part (2).

When either part (2) or part (3) is empty, the function reduces to a KK-MBF.

Note1. A zero vertex can appear only between the leftmost units, and a unit vertex only between the rightmost zeros within a layer – provided that monotonicity is preserved.

Otherwise, the function may not be a nearly KK-MBF. An example is shown in Figure 4.

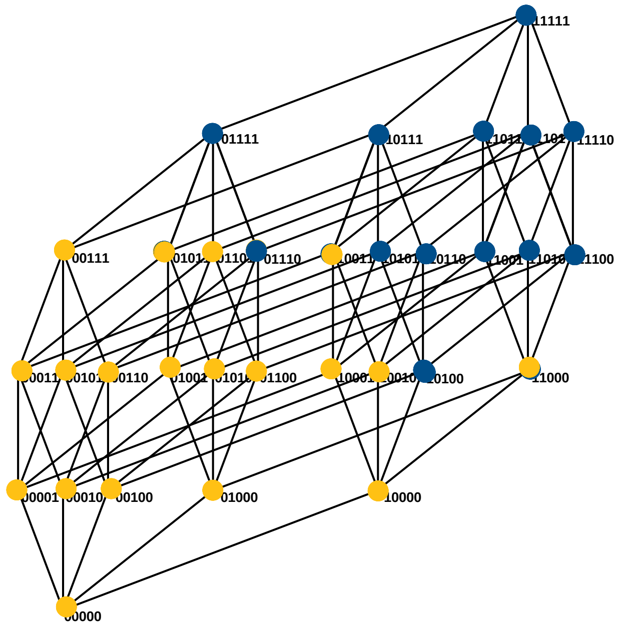

Figure 4.
An example of a nearly KK-MBF function on B^5

Corner points of nearly KK-MBF

Definition 2, which defines the corner points for KK-MBFs, also applies to nearly KK-MBFs.

A layer can contain either a single corner point (a 1-corner or a 0-corner), or two corner points (both a 1-corner and a 0-corner). If a layer contains two corner points, then the 1-corner point is the lexicographic left neighbor of the 0-corner point.

Now, suppose that the parts (2) and/or (3) are not empty.

The following propositions can be established:

Proposition 1. A unit vertex is a 1-corner for a nearly KK-monotone Boolean function if and only if it is the leftmost lower unit of a layer, and it is a 0-corner if and only if it is the rightmost upper zero of a layer.

Proposition 2. The union of the sets of 1-corner and 0-corner points of a nearly KK-MBF is a deadlock-resolving set for the function.

Thus, the union of the sets of 1-corner and 0-corner points forms a resolving set for nearly KK-MBFs. It is also deadlock resolving because the domains of corner points cannot be nested.

V. Concluding Remarks

In this paper, we introduced and studied a natural extension of the KK-MBF class, termed nearly KK-MBF. While KK-MBF functions are characterized by strict initial segments in the lexicographic or reverse-lexicographic order within each layer of the Boolean cube, nearly KK-MBFs relax this structure slightly—allowing a single unit or zero within those segments while still preserving monotonicity. This extension preserves several useful structural properties of KK-MBFs: we showed that the concept of corner points—key to determining the deadlock-resolving set—remains applicable in the nearly KK setting.

In the context of query-based reconstruction of MBFs, where the goal is to reconstruct a hidden function using limited information from observations, cardinality of the sets of corner points (due to their resolving property) will show the minimum number of queries required for the recognition of nearly KK-MBFs, and thus yield complexity estimates for recognizing functions of this class.

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References

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