A Numerical Algorithm for Solving the Non-Linear Version of the Characteristic Problem

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***Abstract*—In this paper, we construct a numerical algorithm to obtain a solution to a characteristic problem posed for one class of hyperbolic quasi-linear PDEs with admissible parabolic degeneration. In the process of constructing our algorithm, we reduce our problem to the numerical analogue of classical Goursat characteristic problem, because the methods of solving of the latter are well studied.**

***Keywords*—Quasi-linear Hyperbolic equation, parabolic degeneracy, characteristic problem, numerical solution.**

I. Introduction

In the paper [1], General integrals were constructed for some equations, which belong to one class of Hyperbolic-Parabolic mixed-type second-order PDEs. It contains squares of the first derivatives of the unknown solution. Both families of characteristics and the set of parabolic degeneracy also depend on the unknown solution. For these equations, we have studied the non-linear variant of the characteristic Goursat problem, and applying the general integral, we proved the existence and uniqueness of a solution to the posed problem. We also found the analytical solutions in implicit form [1].

In the current paper, we consider an analogous problem for the same equation mentioned in [1], and propose an algorithm for the numerical solving of this problem. Equations of this type were considered in papers [2–11], in which initial, characteristic, and initial-characteristic problems were studied.

II. Posing of the Problem

On the two-dimensional space (x, y), we consider a class of non-strictly hyperbolic quasi-linear second-order equations

$$(u_y^2 - u_y)u_{xx} - (2u_x u_y + u_y - u_x - 1)u_{xy} + (u_x^2 + u_x)u_{yy} = \Phi(x, y, u, u_x, u_y), \quad (1)$$

where Φ is a given function of five variables, defined and continuous on the space of independent variables (x, y) and for all finite values of arguments u, u_x, and u_y.

It should be mentioned that posing of characteristic problem for linear equations does not work for non-linear equations. We consider our characteristic problem posed in the following way:

characteristic problem. Suppose we have two Jordan arcs of curves γ and δ derived from the common point (x_0, y_0). They are strictly monotonic, smooth, open, and represented in an explicit way by $y = \varphi(x)$, $y = \psi(x)$, correspondingly. We also assume that $\varphi \in C^2[x_0, a]$, $\psi \in C^2[x_0, b]$. Without losing the Generality, we can assume that $b > a$. Let functions φ, ψ satisfy the conditions:

$$\begin{cases} \varphi\prime(x) \neq 0; -1, x \in [x_0, a], \\ \psi\prime(x) \neq 0; -1, x \in [x_0, b], \\ \varphi(x) \neq \psi(x), x \in (x_0, a], \\ \varphi\prime(x) \neq \psi\prime(x), x \in (x_0, b]. \end{cases} \quad (2)$$

We have to find a solution to the equation (1) when $\Phi = 0$, and the domain of its definition, if we know the value of the solution (x, y), at the point (x_0, y_0), and if the arc γ is a characteristic curve, which belongs to the family of the characteristic root $\lambda_1 = -\frac{u_x+1}{u_y}$ and the arc δ is a characteristic that belongs to the family of the characteristic root $\lambda_2 = -\frac{u_x}{u_y-1}$.

In the case of equation (1), when the right-hand side $\Phi = 0$, the corresponding differential relations have the following form:

$$\begin{cases} (p+1)dx + qdy = 0, \\ (1-q)dp + pdq = 0, \end{cases} \quad (3)$$

and

$$\begin{cases} pdx + (q-1)dy = 0, \\ qdp - (p+1)dq = 0, \end{cases} \quad (4)$$

We add a compatibility equation to these relations

$$du = pdx + qdy. \quad (5)$$

We should note that relations (3), (4), and (5) are equivalent to equation (1). We apply the well-known Monge notations $p \equiv u_x$, $q \equiv u_y$. It also should be noted that (3) corresponds to the root λ_1, while the system (4) corresponds to the root λ_1. These two systems become equivalent only when $p-q+1 = 0$. In this case, the equation (1) degenerates parabolically.

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We consider the case when the following condition is fulfilled on the characteristics φ and ψ

$$p - q + 1 \neq 0. \tag{6}$$

Our goal is to develop and investigate an algorithm for a numerical solution to the posed problem. On the segment $[x_0, a]$, let us introduce a regular grid with a grid size h_1

$$\omega_{h_1} = \{(\widetilde{x}_i^0, \widetilde{y}_i^0), i = 0, 1, ..., n_1, h_1 = \frac{a - \widetilde{x}_0}{n_1},$$

$$\widetilde{x}_i^0 = \widetilde{x}_0 + ih_1, \widetilde{y}_i^0 = \varphi(\widetilde{x}_i^0)\}$$

and on the segment $[x_0, b]$, we introduce the grid with a grid size h_2 :

$$\omega_{h_2} = \{(\widetilde{x}_0^j, \widetilde{y}_0^j), j = 0, 1, ..., n_2, h_2 = \frac{b - x_0}{n_2},$$

$$\widetilde{x}_0^j = \widetilde{x}_0 + jh_2, \widetilde{y}_0^j = \varphi(\widetilde{x}_0^j)\}.$$

Points of ω_{h_1} grid are on the arc γ whereas the points of ω_{h_2} grid are situated on the arc δ. Let us consider the following numerical analogues of differential relations (3) , (4):

$$\begin{cases} (\widetilde{p}_i^j + 1)(\widetilde{x}_{i+1}^j - \widetilde{x}_i^j) + \widetilde{q}_i^j(\widetilde{y}_{i+1}^j - \widetilde{y}_i^j) = 0, \\ (1 - \widetilde{q}_i^j)(\widetilde{p}_{i+1}^j - \widetilde{p}_i^j) + \widetilde{p}_i^j(\widetilde{q}_{i+1}^j - \widetilde{q}_i^j) = 0, \end{cases} \tag{7}$$

$$i = 0, 1, 2, ..., n_1 - 1, j = 0, 1, 2, ..., , n_2.$$

and

$$\begin{cases} \widetilde{p}_i^j(\widetilde{x}_i^{j+1} - \widetilde{x}_i^j) + (\widetilde{q}_i^j - 1)(\widetilde{y}_i^{j+1} - \widetilde{y}_i^j) = 0, \\ \widetilde{q}_i^j(\widetilde{p}_i^{j+1} - \widetilde{p}_i^j) + \widetilde{p}_i^j(\widetilde{q}_i^{j+1} - \widetilde{q}_i^j) = 0, \end{cases} \tag{8}$$

$$j = 0, 1, 2, ..., n_2 - 1, i = 0, 1, 2, ..., , n_1.$$

First, we should find the values of derivatives u_x and u_y at the point (x_0, y_0). For this, in the first equations of both systems, we take $i = j = 0$. Thus, we obtain the linear system with respect to p and q at the point (x_0, y_0).

$$\begin{cases} \widetilde{(p_0^0} + 1)(\widetilde{x}_1^0 - \widetilde{x}_0^0) + (\widetilde{q}_0^0 - 1)(\widetilde{y}_1^0 - \widetilde{y}_0^0) = 0, \\ \widetilde{p}_0^0(\widetilde{x}_0^1 - \widetilde{x}_0^0) + (\widetilde{q}_0^0 - 1)(\widetilde{y}_0^1 - \widetilde{y}_0^1) = 0. \end{cases} \tag{9}$$

Determinant of the system equals to zero when

$$(\widetilde{x}_1^0 - \widetilde{x}_0^0)(\widetilde{y}_0^1 - \widetilde{y}_0^0) - (\widetilde{x}_0^1 - \widetilde{x}_0^0)(\widetilde{y}_1^0 - \widetilde{x}_0^0) = 0,$$

But this is impossible because we have $\lambda_1 \neq \lambda_2$ at the initial point (x_0, y_0) due to the condition (6). Therefore, system (9) has a unique solution.

If we manage to determine the values of $p, q,$ and u at the nodal points of the characteristics γ and δ, it will be possible to reduce our problem to the numerical analogue of the classical Goursat characteristic problem, since the methods of solving it are well studied.

For this purpose, let us consider the following analogue of the system (3) for $j = 0$

$$\begin{cases} (\widetilde{p}_{i+1}^0)(\widetilde{x}_{i+1}^0 - \widetilde{x}_i^0) + \widetilde{q}_{i+1}^0(\widetilde{y}_{i+1}^0 - \widetilde{y}_i^0) = \\ = -(\widetilde{x}_{i+1}^0 - \widetilde{x}_i^0), \\ (1 - \widetilde{q}_i^0)(\widetilde{p}_{i+1}^0 + \widetilde{q}_{i+1}^0 - \widetilde{p}_i^0) = 0, \end{cases} \tag{10}$$

$$i = 0, 1, 2, ..., n_1 - 1.$$

Similarly, in the case of system (4), we can write the corresponding system for $j = 0$

$$\begin{cases} \widetilde{p}_0^{j+1}(\widetilde{x}_0^{j+1} - \widetilde{x}_0^j) + (\widetilde{q}_0^{j+1} - 1)(\widetilde{y}_0^{j+1} - \widetilde{y}_0^j) = 0, \\ \widetilde{q}_0^{j+1}(\widetilde{p}_0^{j+1} - \widetilde{p}_0^j) + (\widetilde{p}_0^{j+1} + 1)(\widetilde{q}_0^{j+1} - \widetilde{q}_0^j) = 0, \end{cases} \tag{11}$$

$$j = 0, 1, 2, ..., n_2 - 1.$$

In (10), we have unknown quantities $p_{i+1}^0, q_{i+1}^0, i = 0, 1, 2, ..., n_1 - 1$, and in (11), the unknown quantities are $p_0^{j+1}, q_0^{j+1}, j = 0, 1, 2, ..., n_2 - 1$.

The systems (10) and (11) are uniquely solvable, because according to (6), the corresponding determinants are different from zero. Thus, the approximate values of the first derivatives at the nodal points of the characteristics φ and ψ are determined uniquely. As for calculating the approximate values of the solution u on the characteristic curves, we consider the following difference analogues of the relation (5):

$$\widetilde{u}_i^0 = \widetilde{u}_{i-1}^0 + \widetilde{p}_{i-1}^0(\widetilde{x}_i^0 - \widetilde{x}_{i-1}^0) + \widetilde{q}_{i-1}^0(\widetilde{y}_i^0 - \widetilde{y}_{i-1}^0), \tag{12}$$

$$i = 1, 2, ..., n_1$$

and

$$\widetilde{u}_0^j = \widetilde{u}_0^{j-1} + \widetilde{p}_0^{j-1}(\widetilde{x}_0^j - \widetilde{x}_0^{j-1}) + \widetilde{q}_0^{j-1}(\widetilde{y}_0^j - \widetilde{y}_0^{j-1}), \tag{13}$$

$$j = 1, 2, ..., n_2.$$

The right-hand sides of both equations consist of known quantities and the values of $\widetilde{u}_i^0$ and $\widetilde{u}_0^j$ can be found in explicit way at the nodal points. Thus, we reduced our problem to the numerical analogue of the classical characteristic problem by using (9)-(13) difference schemes. It is easy to show that the approximation rates of these schemes are $0(h_1)$ and $O(h_2)$, correspondingly.

Let us introduce the following notations: $s_i^j = \widetilde{p}_i^j - p_i^j, g_i^j = \widetilde{q}_i^j - q_i^j, v_i^j = \widetilde{u}_i^j - u_i^j$, where p_i^j, q_i^j and u_i^j are the values of u_x, u_y and u at the grid nodal (i, j) points.

Then, applying (9)-(13) schemes and taking into account the smoothness of the exact solution, we obtain the following relations:

$$\begin{cases} (s_{i+1}^0 + p_{i+1}^0)h_1 + (g_{i+1}^0 + q_{i+1}^0)\varphi'_{i+1}h_1 + \\ +h_1^2 = -h_1, \\ (1 - (g_i^0 + q_i^0))(s_{i+1}^0 + p_{i+1}^0) = \\ (1 - (g_{i+1}^0 + q_{i+1}^0))(s_i^0 + p_i^0), \end{cases} \tag{14}$$

$$i = 0, 1, 2, ..., n_1 - 1.$$

$$\begin{cases} (s_0^{j+1} + p_0^{j+1})h_2 + \\ +(g_0^{j+1} + q_0^{j+1} - 1)\psi'^{j+1}_0 h_2 = 0, \\ (g_0^j + q_0^j)(s_0^{j+1} + p_0^{j+1} - s_0^j - p_0^j) = \\ (s_0^j + p_0^j + 1)(g_0^{j+1} + q_0^{j+1} - g_0^j - q_0^j), \end{cases} \tag{15}$$

$$j = 0, 1, 2, ..., n_2 - 1.$$

$$v_i^0 + u_i^0 = v_{i-1}^0 + u_{i-1}^0 + (p_{i-1}^0 + \\ +s_{i-1}^0)h_1 + (q_{i-1}^0 + g_{i-1}^0)\varphi_i' h_1 \tag{16}$$

$$i = 1, 2, ..., n_1.$$

$$v_0^j + u_0^j = v_0^{j-1} + u_0^{j-1} + (p_0^{j-1} + s_0^{j-1})h_2 + \\ +(q_0^{j-1} + g_0^{j-1})\psi'^j_0 h_2, \tag{17}$$

$$j = 1, 2, ..., n_2 - 1.$$

Let us consider the equations (14) and (15). From these equations, taking into account the error of approximation, we can obtain:

$$s_{i+1}^0 + g_{i+1}^0 \varphi_i' = O(h_1^2) \tag{18}$$

$$(1 - g_i^0 - q_i^0)s_{i+1}^0 + (g_{i+1}^0 + q_{i+1}^0 - 1)s_i^0 + \\ +p_{i+1}^0 g_i^0 + g_{i+1}^0 p_i^0 = O(h_1^2), \tag{19}$$

$$i = 1, 2, ..., n_1 - 1.$$

From the equation (18), we can express s_{i+1}^0 and put it into (19). Also taking into account the second equation of (7), we obtain:

$$g_{i+1}^0 = -\frac{1}{p_i^0}(p_{i+1}^0 + (1 - q_{i+1}^0))g_i^0 + O(h_1),$$

$$p_i^0 \neq 0, i = 0, 1, 2, ..., n_1 - 1.$$

If we write a similar relation for g_i^0 and put it into the last relation, we can express g_{i+1}^0 by g_i^0 :

$$g_{i+1}^0 = O(h_1) - g_0^0 = O(h_1),$$

Because

$$g_0^0 = \widetilde{g}_0^0 - q_0^0$$

Thus, we can write:

$$||g_{h_1}||_C \leq Ch_1, \ C = const.$$

Taking into account this inequality, from (18), it is easy to obtain the following estimation:

$$||s_{h_1}||_C = O(h_1).$$

The convergence of the difference scheme (11), (12), (13) is proved analogously.

Thus, we proved the following

Theorem 1: If the solution of the characteristic problem for (1) and the functions φ and ψ satisfy the following conditions

$$u \in C^3[x_0, b], \varphi \in C^2[x_0, a], \psi \in C^2[x_0, b], b > a$$

and conditions (2) and (6) hold, and also in the domain of definition of the solution we have $u_x \neq 0$, then the difference scheme (11), (13) converges and

$$||\widetilde{p} - p||_C = ||\widetilde{q} - q||_C = ||\widetilde{u} - u||_C = O(h),$$

where $h = \max(h_1, h_2)$.

As we mentioned before, the difference scheme (10)-(13) makes it possible to reduce the characteristic problem 1 to the difference analogue of the classical Goursat problem, since we already have the approximate values ofu, u_x, u_y at the nodal points of the characteristics γ and δ. Consequently, we can use the difference schemes (7), (8), and this way we obtain the approximate values of the solution for our problem. In the process of solving the problem, we also construct the domain of definition of the solution. We do not prove here the convergence of the schemes (7), (8), because the proof is absolutely similar to the one given in [4].

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References

[1] G.G. Baghaturia and M.Z. Menteshashvili, “A General Integral of a Quasilinear Equation and Application to a Nonlinear Characteristic Problem”, *Siberian Mathematical Journal*, vol. 60, Issue 6, pp. 940 – 9511, 2019.

[2] G.G. Baghaturia, ”Nonlinear versions of hyperbolic problems for one quasi-linear equation of mixed type”, *J. Math. Sci.*, vol.208, no. 6, pp.621–634, 2015.

[3] M.Z. Menteshashvili, “The Nonlinear Cauchy Problem with Solutions Defined in Domains with Gaps , *Journal of Mathematical Sciences (United States)*, vol. 206, Issue 4, pp. 413 – 4231, 2015.

[4] G.G. Baghaturia and M.Z. Menteshashvili, “On the Numerical Solution of the Characteristic Problems for One Class of Quasilinear Equations”, *Program Comput Soft*, 50 (Suppl 1), pp. S33–S38, 2024.

[5] A.V. Favorskaya and I.B. Petrov, ”Grid-characteristic method, in Innovations in Wave Processes Modelling and Decision Making”. *Smart Innovation, Systems and Technologies*, vol. 90, Springer, 2018.

[6] V. Kvaratskhelia and M. Menteshashvili, ”Some Nonlinear Version of a Nonlocal Problem and Its Discrete Analogy,” *Computer Science and Information Technologies (CSIT)*, Yerevan, Armenia, pp. 77–78, 2019.

[7] M. Menteshashvili,”Study of the nonlinear characteristic problem using the Riemann invariants”, *Proc. A. Razmadze Math. Inst.*, vol. 160, pp. 113–120, 2012.

[8] G. Baghaturia and M. Menteshashvili, ”Application of general integral of quasilinear equation to solving of non-linear Cauchy problem”, *Bull. TICMI*, vol. 27, no. 2, pp. 59–65, 2023.

[9] J. Gvazava, G. Baghaturia and M. Menteshashvili. ”Cauchy problem for a quasilinear hyperbolic equation with closed support of data”, *J. Math. Sci.*, vol. 193, no. 3, pp. 364–368, 2013.

[10] R. Bitsadze and M. Menteshashvili. ”On one nonlinear variant of the nonlocal characteristic problem”, *J. Math. Sci.*, vol. 206, no. 4, pp. 406–412, 2015.

[11] G. Baghaturia. ”Some non-linear versions of hyperbolic problems for one quasilinear equation of mixed type”, *J. Math. Sci.*, vol. 208, no. 6, pp. 621–634, 2015.